QM math structure - Problem Set 7

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1 a linear chain with N mass

$$L = \frac{m}{2}\sum_r \dot{x}_r^2 - \frac{\alpha}{2}\sum(x_{r+1} - x_r - a)^2 - \frac{\beta}{2}\sum_r (x_r - x_r^0)^2$$

(a) $q_r = x_r - x_r^0$, where $x_r^0 = ra = r\frac{l}{N}$, $a = \frac{l}{N}$. Therefore,

$$\begin{cases} \dot{q}_r & = \dot{x}_r \\ x_{r+1} - x_r - a & = [q_{r+1} + (r+1)a] - [q_r + ra] - a = q_{r+1} - q_r \end{cases}$$

$\Rightarrow$

$$L = \frac{m}{2}\sum_r \dot{q}_r^2 - \frac{\alpha}{2}\sum(q_{r+1} - q_r)^2 - \frac{\beta}{2}\sum_r (q_r)^2$$

(b) Define $q_r(t) = \sqrt{a/m}\varphi(x_r^0, t)$. In the limit $\alpha, N \to \infty$ and $\beta, m \to 0$ such that $\alpha a \to T$, $m/a \to \mu$, and

$$\frac{\beta a}{m} \to \frac{T}{\mu}\frac{1}{\lambda_c^2}$$

Thus, $a \to 0$ and use this we can get

$$\varphi(x_r^0, t) = \varphi(x, t)$$

$$\varphi(x_{r+1}^0, t) = \varphi(x_r^0, t) + a\left(\frac{\partial\varphi}{\partial x}\right)|_{x=x_r^0} + \cdots$$

$$\lim_{a\to 0}\frac{\varphi(x_{r+1}^0, t) - \varphi(x_r^0, t)}{a} = \left(\frac{\partial\varphi}{\partial x}\right)|_{x=x_r^0}$$

$$\sum_r a = \int dx$$

Therefore,

$$\begin{aligned} L &= \tfrac{m}{2}\textstyle\sum_r \dot{q}_r^2 - \tfrac{\alpha}{2}\sum(q_{r+1} - q_r)^2 - \tfrac{\beta}{2}\sum_r (q_r)^2 \\ &= \tfrac{m}{2}\textstyle\sum_r (\sqrt{a/m})^2(\tfrac{\partial\varphi}{\partial t})^2 - \tfrac{\alpha a}{2m/a}\sum_r a(\tfrac{\varphi(x_{r+1}^0,t) - \varphi(x_r^0,t)}{a})^2 - \tfrac{\beta a}{2m}\varphi(x,t)^2 \\ &= \textstyle\sum a\tfrac{1}{2}\left\{\left(\tfrac{\partial\varphi}{\partial t}\right)^2 - \tfrac{T}{\mu}\left(\tfrac{\partial\varphi}{\partial x}\right)^2 - \tfrac{T}{\mu}\tfrac{1}{\lambda_c^2}\varphi^2\right\} \\ &= \textstyle\int dx a\tfrac{1}{2}\left\{\left(\tfrac{\partial\varphi}{\partial t}\right)^2 - \tfrac{T}{\mu}\left(\tfrac{\partial\varphi}{\partial x}\right)^2 - \tfrac{T}{\mu}\tfrac{1}{\lambda_c^2}\varphi^2\right\} = \int dx\mathcal{L}, \end{aligned}$$

where the Lagrangian density is defined as

$$\mathcal{L} = \frac{1}{2}\left\{\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{T}{\mu}\left(\frac{\partial\varphi}{\partial x}\right)^2 - \frac{T}{\mu}\frac{1}{\lambda_c^2}\varphi^2\right\}$$

such that

$$L = \int dx\mathcal{L}$$

The action

$$A = \int dtL = \int dtdx\mathcal{L}(\varphi, \dot{\varphi}, \varphi')$$

is obtained.

2 Verification

$$q_r = \sqrt{\frac{\hbar}{Nm}}\sum_s \exp\left(\frac{2\pi i}{N}sr\right)Q_s$$

Then,

$$q_{r+1} - q_r = \sqrt{\frac{\hbar}{Nm}}\sum_s \left[\exp\left(\frac{2\pi i}{N}s\right) - 1\right]\exp\left(\frac{2\pi i}{N}sr\right)Q_s$$

Because $q_r^\dagger = q_r$ which is real (leading to $Q_{-s} = Q_s^\dagger$),

$$q_{r+1} - q_r = \sqrt{\frac{\hbar}{Nm}}\sum_s \left[\exp\left(\frac{-2\pi i}{N}s\right) - 1\right]\exp\left(\frac{-2\pi i}{N}sr\right)Q_s^\dagger$$

Therefore,

$$\begin{aligned}
&\textstyle\sum_r (q_{r+1} - q_r)^2 \\
&= \tfrac{\hbar}{m}\textstyle\sum_{ss'}\left[\exp\left(\frac{2\pi i}{N}s\right) - 1\right]\left[\exp\left(\frac{-2\pi i}{N}s'\right) - 1\right]\sum_r \frac{1}{N}\exp\left(\frac{2\pi i}{N}r(s-s')\right)Q_sQ_{s'}^\dagger \\
&= \tfrac{\hbar}{m}\textstyle\sum_{ss'}\left[\exp\left(\frac{2\pi i}{N}s\right) - 1\right]\left[\exp\left(\frac{-2\pi i}{N}s'\right) - 1\right]\delta_{ss'}Q_sQ_{s'}^\dagger \\
&= \tfrac{\hbar}{m}\textstyle\sum_s(-4)\left(\frac{1}{2i}\right)^2\left[\exp\left(\frac{\pi i}{N}s\right) - \exp\left(\frac{-\pi i}{N}s\right)\right]\left[\exp\left(\frac{-\pi i}{N}s'\right) - \exp\left(\frac{\pi i}{N}s'\right)\right]Q_sQ_s^\dagger \\
&= \tfrac{4\hbar}{m}\textstyle\sum_s \sin^2\left(\frac{\pi s}{N}\right)Q_sQ_s^\dagger
\end{aligned}$$

As required.

3 Verification

The normal coordinate Q_s and momentum P_s are introduced by Fourier expansion

$$q_r = \sqrt{\frac{\hbar}{Nm}} \sum_s e^{\frac{2\pi i}{N} rs} Q_s$$

$$p_r = \sqrt{\frac{m\hbar}{N}} \sum_s e^{\frac{2\pi i}{N} rs} P_s$$

Thus,

$$Q_s = \sqrt{\frac{m}{N\hbar}} \sum_r e^{-\frac{2\pi i}{N} rs} q_r$$

$$P_s = \frac{1}{\sqrt{mN\hbar}} \sum_r e^{-\frac{2\pi i}{N} rs} p_r$$

Because of the commutations between q_r and p_r, we can get the commutations between Q_s and P_s as the following:

$$[Q_s, Q_{s'}] = [P_s, P_{s'}] = 0$$

$$\begin{aligned} [Q_s, P_{s'}] &= \tfrac{1}{N\hbar} \textstyle\sum_{rr'} [q_r, p_{r'}] e^{-\frac{2\pi i}{N}(rs+r's')} \\ &= \tfrac{i}{N} \textstyle\sum_r e^{-\frac{2\pi i}{N} r(s+s')} \\ &= i\delta_{s+s',0} = i\delta_{s,-s'} \end{aligned}$$

According to the eqution of motion in the Heisenberg picture, we can further expand Q_s an P_s into

$$Q_s = \frac{1}{\sqrt{2\omega(s)}} \left(a_s e^{-i\omega(s)t} + a_{-s}^\dagger e^{i\omega(s)t} \right)$$

$$P_s = i\sqrt{\frac{\omega(s)}{2}} \left(-a_s e^{-i\omega(s)t} + a_{-s}^\dagger e^{i\omega(s)t} \right)$$

The coefficients are chosen such that

$$Q_s^\dagger = \frac{1}{\sqrt{2\omega(s)}} \left(a_s^\dagger e^{i\omega(s)t} + a_{-s} e^{-i\omega(s)t} \right) = Q_{-s}$$

Similarly for $P_s^\dagger = P_{-s}$. Define

$$b_s = a_s e^{-i\omega(s)t} \text{ and } b_{-s}^\dagger = a_{-s}^\dagger e^{i\omega(s)t}$$

Thus, we can rewrite the normal coordinate and the normal momentum as follows

$$Q_s = \frac{1}{\sqrt{2\omega(s)}}(b_s + b_{-s}^\dagger)$$

$$P_s = i\sqrt{\frac{\omega(s)}{2}}(-b_s + b_{-s}^\dagger)$$

Therefore,

$$b_s = \sqrt{\frac{\omega(s)}{2}}Q_s + i\sqrt{\frac{1}{2\omega(s)}}P_s$$

$$b_{-s}^\dagger = \sqrt{\frac{\omega(s)}{2}}Q_s - i\sqrt{\frac{1}{2\omega(s)}}P_s$$

Note that $\omega(s) = \omega(-s)$ is an even function. Therefore,

$$b_s^\dagger = \sqrt{\frac{\omega(s)}{2}}Q_{-s} - i\sqrt{\frac{1}{2\omega(s)}}P_{-s}$$

Next step is to make use of the commutations previously derived:

$$[Q_s, Q_{s'}] = [P_s, P_{s'}] = 0,\ [Q_s, P_{s'}] = i\delta_{s,-s'}I$$

Then we know,

$$[b_s, b_{s'}] = \frac{i}{2}[Q_s, P_{s'}] + \frac{i}{2}[P_s, Q_{s'}] = 0$$

$$[b_s^\dagger, b_{s'}^\dagger] = \frac{-i}{2}[Q_{-s}, P_{-s'}] - \frac{i}{2}[P_{-s}, Q_{-s'}] = 0$$

$$[b_s, b_{s'}^\dagger] = \frac{-i}{2}[Q_s, P_{-s'}] + \frac{i}{2}[P_s, Q_{-s'}] = \frac{-i}{2}i\delta_{s,s'} + \frac{i}{2}(-i)\delta_{s,s'} = \delta_{s,s'}I$$

Because of

$$a_s = b_s e^{i\omega(s)t} \text{ and } a_{-s}^\dagger = b_{-s}^\dagger e^{-i\omega(s)t},$$

we know

$$[a_s, a_{s'}] = [a_s^\dagger,^\dagger_{s'}] = 0$$

$$[a_s, a_{s'}^\dagger] = e^{i\omega(s)t - i\omega(s')t}[b_s, b_{s'}^\dagger] = e^{it(\omega(s) - i\omega(s'))}\delta_{s,s'}I = \delta_{s,s'}I$$