Problem Set 7 (Due: Nov. 9, 2011)

November 8, 2011

Your homework should be submitted to TA (Chih-Han Lin) at the beginning of the course. Generally, You have a week to finish the problem set since it was revealed. If you cannot submit it on time, you can still bring your homework to TA's office (L325, Institute of Atomic and Molecular Sciences, in NTU campus) and put it into TA's mailbox which is on the top of the shoe cabinet in L325. Always remember to remind TA to check your homework by e-mail (`clin@ltl.iams.sinica.edu.tw`) if you don't submit it in class.

Ex 1

For the system of a linear chain with N mass point of m equal spacing in the lengh l. Each mass point is attached by a first spring of force constant α to the nearest neighbor mass points, and by the second spring of force constant β to the equilibrium positions $x_r^0 = rl/N = ra$. Then the Lagrangian of the system can be expressed as

$$\mathcal{L} = \frac{m}{2}\sum_r \dot{x_r}^2 - \frac{\alpha}{2}\sum (x_{r+1} - x_r - a)^2 - \frac{\beta}{2}\sum_r (x_r - x_r^0)^2. \tag{1}$$

(a) Define the displace of r-th mass point from its equilibrium position by q_r, then

$$\mathcal{L} = \frac{m}{2}\sum_r \dot{q_r}^2 - \frac{\alpha}{2}\sum (q_{r+1} - q_r)^2 - \frac{\beta}{2}\sum_r (q_r)^2. \tag{2}$$

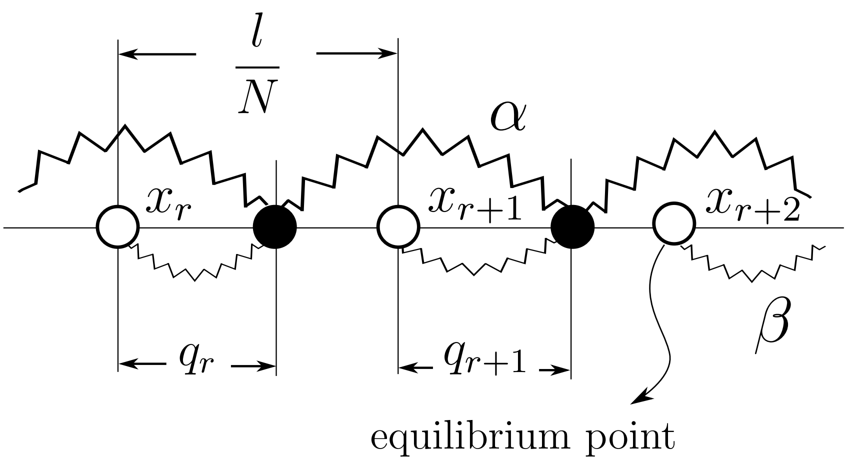

Figure 1: figure of ex.1

(b) Define $\varphi(x_r^0, t)$ as a new dynamical variable such that $q_r(t) = \sqrt{a/m}\varphi(x_r^0, t)$ then show that the action of the system A can be written as

$$A = \int dtdx \frac{1}{2}\left\{\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{T}{\mu}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{T}{\mu}\frac{1}{\lambda_c}\varphi^2\right\}. \tag{3}$$

in the limit $\alpha, N \longrightarrow \infty$ and $\beta, m \longrightarrow 0$ such that $\alpha a \longrightarrow T$, $m/a \longrightarrow \mu$ and

$$\frac{\beta a}{m} \longrightarrow \frac{T}{\mu}\frac{1}{\lambda_c^2}.$$

where $A = \int dtdx\mathcal{L}(\varphi, \dot{\varphi}, \varphi')$ with

$$\mathcal{L} = \frac{1}{2}(\dot{\varphi} - \frac{T}{\mu}\varphi'^2 - \frac{T}{\mu}\frac{1}{\lambda_c^2}\varphi^2).$$

Ex 2

Verify that

$$\sum_r (\boldsymbol{q}_r - \boldsymbol{q}_{r+1})^2 = \frac{\hbar}{m}\sum_s 4\sin^2\left(\frac{2\pi s}{N}\right)\boldsymbol{Q}_s\boldsymbol{Q}_s^\dagger, \tag{4}$$

if

$$\boldsymbol{q}_r = \sqrt{\frac{\hbar}{mN}}\sum_s \exp\left(\frac{2\pi i}{N}sr\right)\boldsymbol{Q}_s.$$

Ex 3

Verify that

$$[\boldsymbol{a}_s, \boldsymbol{a}_{s'}] = [\boldsymbol{a}_s^\dagger, \boldsymbol{a}_{s'}^\dagger] = 0,$$
$$[\boldsymbol{a}_s, \boldsymbol{a}_{s'}^\dagger] = \delta_{ss'}\mathbf{I}.$$