Problem Set 3 (Due: Oct. 12,2011)

October 5, 2011

Your homework should be submitted to TA (Chih-Han Lin) at the beginning of the course. Generally, You have a week to finish the problem set since it was revealed. If you cannot submit it on time, you can still bring your homework to TA's office (L325, Institute of Atomic and Molecular Sciences, in NTU campus) and put it into TA's mailbox which is on the top of the shoe cabinet in L325. Always remember to remind TA to check your homework by e-mail (`clin@ltl.iams.sinica.edu.tw`) if you don't submit it in class.

Ex 1

Prove that if $\mathbf{A}$ and $\mathbf{B}$ are two operators that both commute with their commutator, then

$$e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{A}+\mathbf{B}+[\mathbf{A},\mathbf{B}]/2}.$$

Ex 2

If two observables $\mathbf{A}_1$ and $\mathbf{A}_2$ are not compatible, but their corresponding operators both commute with the Hamiltonian operator $\mathbf{H}$, i.e.

$$[\mathbf{A}_1, \mathbf{H}] = [\mathbf{A}_2, \mathbf{H}] = 0.$$

Show that the energy eigenstates are in general degenerate.

Ex 3

Consider a one-dimensional Hamiltonian

$$\mathbf{H} = \frac{1}{2m}\mathbf{P}^2 + V(\mathbf{Q})$$

and use the fact that the commutator of $\mathbf{Q}$ and $[\mathbf{Q}, \mathbf{H}]$ is a constant operator, to show that

$$\sum_k (E_k - E_s)|Q_{sk}|^2 = \frac{\hbar^2}{2m},$$

which is referred to as the Thomas-Reiche-Kuhn sum rule, where $Q_{sk} = (\psi_s, \mathbf{Q}\psi_k)$ and ψ_s is the eigenstate of $\mathbf{H}$ with eigenvalue E_s, i.e. $\mathbf{H}\psi_s = E_s\psi_s$.

Ex 4

Prove that if

$$\mathbf{U}(L_3, \theta) = \exp\left[\frac{i}{\hbar}(XP_y - YP_x)\right],$$

then

$$\mathbf{U}X\mathbf{U}^{-1} = X\cos\theta - Y\sin\theta,$$
$$\mathbf{U}Y\mathbf{U}^{-1} = X\sin\theta + Y\cos\theta.$$

Ex 5

Show that if

$$\mathbf{U}(\vec{L}, \vec{\theta}) = \exp\left(\frac{i}{\hbar}\vec{\theta}\cdot\vec{L}\right),$$

then $\mathbf{U}$ commutes with $\vec{X}\cdot\vec{X}$, $\vec{Y}\cdot\vec{Y}$ and $\vec{Z}\cdot\vec{Z}$. (**Hint:** Try to show that $\mathbf{U}[X^2 + Y^2 + Z^2]\mathbf{U}^{-1} = X^2 + Y^2 + Z^2$.)