# Problem Set 11 (Due: Dec. 14, 2011) 

December 7, 2011

Your homework should be submitted to TA (Chih-Han Lin) at the beginning of the course. Generally, You have a week to finish the problem set since it was revealed. If you cannot submit it on time, you can still bring your homework to TA's office (L325, Institute of Atomic and Molecular Sciences, in NTU campus) and put it into TA's mailbox which is on the top of the shoe cabinet in L325. Always remember to remind TA to check your homework by e-mail (clin@ltl.iams.sinica.edu.tw) if you don't submit it in class.

## Ex 1

Consider a $2 \times 2$ matrix $\mathbf{A}$ for the linear transformation in 2-dimensional vector space, i.e.

$$
\binom{x^{\prime}}{y^{\prime}}=\mathbf{A}\binom{x}{y}
$$

such that the transformation leaves $x^{2}-y^{2}$ invariant, namely $x^{2}-y^{2}=x^{\prime 2}-y^{\prime 2}$.
(a) Show that A satisfied

$$
\mathbf{A}^{T}\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) \mathbf{A}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

(b) Show that the transformation forms a group called $\mathrm{O}(1,1)$-group.
(c) Construct explicitly the matrix $\mathbf{A}$ with $\operatorname{det} \mathbf{A}=1$.
(d) Construct explicitly the generator of $\mathbf{A}$ and reconstruct $\mathbf{A}$ through its generator.

## Ex 2

Consider the following transformation,

$$
x^{\prime}=a^{1} x+a^{2} .
$$

(a) What are the conditions if the transformation forms a 2 parameters group?
(b) Denote the group element by $a=\left(a^{1}, a^{2}\right)$. Find the inverse element of $a$, namely, $a^{-1}=\left(\left(a^{-1}\right)^{1},\left(a^{-1}\right)^{2}\right)$.
(c) What are the composition rules of the group parameters, i.e. try to find $c=b a$ for $c^{l}=c^{l}\left(a^{i}, b^{j}\right)=f^{l}\left(a^{i}, b^{j}\right) ?$
(d) Can you construct the $2 \times 2$ matrix representation of the group to justify your answer in (a), (b) and (c)?

## Ex 3

Consider the rotational transformation about the azimuthal axis with an angle $\theta$, namely,

$$
\begin{aligned}
& x^{\prime 1}=x^{1} \cos \theta-x^{2} \sin \theta, \\
& x^{\prime 2}=x^{1} \sin \theta+x^{2} \cos \theta, \\
& x^{\prime 3}=x^{3},
\end{aligned}
$$

then the function $F(x)=F\left(x^{1}, x^{2}, x^{3}\right)$ will be changed into

$$
F\left(x^{\prime}\right)=F\left(x^{1} \cos \theta-x^{2} \sin \theta, x^{1} \sin \theta+x^{2} \cos \theta, x^{3}\right)
$$

Prove explicitly that the transformed function can be obtained by

$$
\begin{aligned}
\exp \left(\theta X_{3}\right) F(x) & =\exp \left[\theta\left(-x^{2} \frac{\partial}{\partial x^{1}}+x^{1} \frac{\partial}{\partial x^{2}}\right)\right] F\left(x^{1}, x^{2}, x^{3}\right) \\
& =F\left(x^{\prime 1}, x^{\prime 2}, x^{\prime 3}\right)
\end{aligned}
$$

(Hint: use $\exp \left(\theta X_{3}\right)=\exp (\theta \partial / \partial \varphi)$ and write

$$
\left.F(x)=F\left(r \cos \varphi, r \sin \varphi, x^{3}\right) .\right)
$$

