

Mathematical Structure Problem Set 3
of

Quantum Mechanics (I)

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$$1. f(\lambda) = e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}$$

$$\Rightarrow \frac{df}{d\lambda} = Ae^{\lambda A}e^{\lambda B}e^{-\lambda(A+B)} + e^{\lambda A}Be^{\lambda B}e^{-\lambda(A+B)} + e^{\lambda A}e^{\lambda B}(-A-B)e^{-\lambda(A+B)}$$

$$= Ae^{\lambda A}e^{\lambda B}e^{-\lambda(A+B)} + \cancel{e^{\lambda A}Be^{\lambda B}e^{-\lambda(A+B)}} - \cancel{e^{\lambda A}e^{\lambda B}Ae^{-\lambda(A+B)}} - \cancel{e^{\lambda A}e^{\lambda B}Be^{-\lambda(A+B)}}$$

(∵ $IBe^{\lambda B} = e^{\lambda B}IB$)

$$= Ae^{\lambda A}e^{\lambda B}e^{-\lambda(A+B)} - e^{\lambda A}e^{\lambda B}Ae^{-\lambda(A+B)}$$

$$\therefore e^{\lambda A}e^{\lambda B}Ae^{-\lambda(A+B)}$$

$$= e^{\lambda A} \underbrace{e^{\lambda B}Ae^{-\lambda B}}_{=0} e^{\lambda B}e^{-\lambda(A+B)}$$

$$= e^{\lambda A} \left(A + \lambda [B, A] + \frac{\lambda^2}{2} [B, [B, A]] + \dots \right) e^{\lambda B}e^{-\lambda(A+B)}$$

$$= Ae^{\lambda A} + \lambda e^{\lambda A} [B, A] e^{\lambda B}e^{-\lambda(A+B)}$$

$$\therefore \frac{df}{d\lambda} = \lambda e^{\lambda A} [A, B] e^{\lambda B}e^{-\lambda(A+B)}$$

$$= \lambda e^{\lambda A} [A, B] \underbrace{e^{-\lambda A}e^{\lambda A}}_{=0} e^{\lambda B}e^{-\lambda(A+B)}$$

$$= \lambda \left([A, B] + \lambda [A, [A, B]] + \frac{\lambda^2}{2!} [A, [A, [A, B]]] + \dots \right) f(\lambda)$$

$$= \lambda [A, B] f(\lambda)$$

By integrating both sides, $f(\lambda) = \exp\left[\frac{\lambda^2}{2}[A, B]\right]$,

$$\text{let } \lambda = 1, \text{ then } e^A e^B e^{-(A+B)} = e^{\frac{1}{2}[A, B]}$$

$$\Rightarrow e^A e^B = e^{\frac{1}{2}[A, B]} e^{A+B}$$

$$\therefore [A+B, [A, B]] = [A, [A, B]] + [B, [A, B]] = 0 + 0 = 0$$

$\Rightarrow A+B$ and $[A, B]$ are commute.

$$\therefore e^A e^B = e^{\frac{1}{2}[A, B]} e^{A+B} = e^{A+B+\frac{1}{2}[A, B]} \quad \text{※}$$

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$$\begin{aligned} 3. \quad & \langle \psi_s | [Q, [Q, H]] | \psi_s \rangle \\ &= \langle \psi_s | [Q, QH - HQ] | \psi_s \rangle \\ &= \langle \psi_s | (QQH - 2QHQ + HQQ) | \psi_s \rangle \end{aligned}$$

* (如果助教覺得不能用 Dirac notation,
最後一頁有內積的寫法，這頁只是太順手
就用了 Dirac notation)

$$\begin{aligned} \langle \psi_s | QHQ | \psi_s \rangle &= \langle \psi_s | QQ E_s | \psi_s \rangle = E_s \langle \psi_s | QQ | \psi_s \rangle = \\ &= \sum_k E_k \langle \psi_s | Q | \psi_k \rangle \langle \psi_k | Q | \psi_s \rangle \\ &= \sum_k E_k | Q_{sk} |^2. \end{aligned}$$

$$\langle \psi_s | HQQ | \psi_s \rangle = E_s \langle \psi_s | QQ | \psi_s \rangle = \sum_k E_k | Q_{sk} |^2$$

$$\begin{aligned} \langle \psi_s | QHQ | \psi_s \rangle &= \sum_i \sum_j \langle \psi_s | Q | \psi_i \rangle \langle \psi_i | H | \psi_j \rangle \langle \psi_j | Q | \psi_s \rangle \\ &= \sum_i \sum_j Q_{is} \cdot E_i \cdot \delta_{ij} \cdot Q_{sj} = \sum_j E_j | Q_{sj} |^2 = \sum_k E_k | Q_{sk} |^2 \end{aligned}$$

$$\therefore \langle \psi_s | [Q, [Q, H]] | \psi_s \rangle = 2 \left(\sum_k (E_s - E_k) | Q_{sk} |^2 \right)$$

$$\begin{aligned} [Q, [Q, H]] &= [Q, [Q, \frac{P^2}{2m} + V(Q)]] = [Q, \frac{1}{2m} [Q, P^2]] \\ &= [Q, \frac{1}{2m} \cdot 2 \cdot i\hbar p] = i\hbar \cdot \frac{1}{m} [Q, p] = -\frac{\hbar^2}{m} \end{aligned}$$

$$\Rightarrow 2 \sum_k (E_s - E_k) | Q_{sk} |^2 = \langle \psi_s | [Q, [Q, H]] | \psi_s \rangle = -\frac{\hbar^2}{m}$$

$$\Rightarrow \sum_k (E_k - E_s) | Q_{sk} |^2 = \frac{\hbar^2}{m} \quad \text{※}$$

$$4. \quad \because e^{\lambda A} B e^{-\lambda A} = B + \lambda [A, B] + \frac{\lambda^2}{2!} [A, [A, B]] + \frac{\lambda^3}{3!} [A, [A, [A, B]]] \\ /0 \quad \quad \quad + \frac{\lambda^4}{4!} [A, [A, [A, [A, B]]]] + \dots$$

$$\text{Let } \lambda = \frac{i\theta}{\hbar}, \quad A = xP_y - yP_x, \quad B = x,$$

$$\text{then } [A, B] = [xP_y - yP_x, x] = [xP_y, x] - [yP_x, x]$$

$$= x[P_y, x] + [x, x]P_y - \underline{y[P_x, x]} - [y, x]P_x \\ = i\hbar y.$$

$$[A, [A, B]] = [xP_y - yP_x, i\hbar y]$$

$$= i\hbar ([xP_y, y] - [yP_x, y]) = -i\hbar \cdot (-i\hbar x)$$

$$[A, [A, [A, B]]] = - (i\hbar)^2 [xP_y - yP_x, x] = - (i\hbar)^3 y$$

⋮

$$\therefore \exp \frac{i\theta}{\hbar} (xP_y - yP_x) x e^{-\frac{i\theta}{\hbar} (xP_y - yP_x)}$$

$$= x + \left(\frac{i\theta}{\hbar}\right) \cdot (i\hbar y) + \frac{1}{2!} \left(\frac{i\theta}{\hbar}\right)^2 \cdot (-)(i\hbar)^2 x + \frac{1}{3!} \left(\frac{i\theta}{\hbar}\right)^3 \cdot (-)(i\hbar)^3 y + \dots$$

$$= x - \theta y - \frac{1}{2!} \theta^2 x + \frac{1}{3!} \theta^3 y + \dots$$

$$= x \left(1 - \frac{\theta^2}{2!} + \dots\right) - y \left(\theta - \frac{1}{3!} \theta^3 + \dots\right) = x \cos \theta - y \sin \theta x.$$

Similarly, let $C = y$.

~~$$\text{then } [A, C] = [xP_y - yP_x, y] = -i\hbar x$$~~

$$[A, [A, C]] = (-i\hbar) [A, x] = - (i\hbar)^2 y$$

$$[A, [A, [A, C]]] = (i\hbar)^3 x$$

$$\begin{aligned}
 & \therefore e^{\frac{i\theta}{\hbar}(xP_y - yP_x)} y e^{-\frac{i\theta}{\hbar}(xP_y - yP_x)} \\
 &= y + \left(\frac{i\theta}{\hbar}\right)(-i\hbar)x + \frac{1}{2!} \left(\frac{i\theta}{\hbar}\right)^2 (-i\hbar)^2 y + \frac{1}{3!} \left(\frac{i\theta}{\hbar}\right)^3 (-i\hbar)^3 x + \dots \\
 &= y \left(1 - \frac{\theta^2}{2!} + \dots\right) + x \left(\theta - \frac{\theta^3}{3!} + \dots\right) = y \cos \theta + x \sin \theta \quad \times
 \end{aligned}$$

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 By proposition 7 in the text book, there exists a complete set of states which are simultaneously eigenvectors of A_1 and H .

Let ψ_{α, a_1} be these eigenvectors such that $\begin{cases} A_1 \psi_{\alpha, a_1} = a_1 \psi_{\alpha, a_1} \\ H \psi_{\alpha, a_1} = E_\alpha \psi_{\alpha, a_1} \end{cases}$

Similarly, there exists simultaneously eigenvectors of A_2 and H .

Let ϕ_{α, a_2} be these eigenvectors such that $\begin{cases} A_2 \phi_{\alpha, a_2} = a_2 \phi_{\alpha, a_2} \\ H \phi_{\alpha, a_2} = E_\alpha \phi_{\alpha, a_2} \end{cases}$

If the energy eigenstates are non-degenerate, then for any eigenvalue α of H , there is only one eigenvector corresponding to it.

So that $\psi_{\alpha, a_1} = C \cdot \phi_{\alpha, a_2}$, C is constant.

$$\therefore A_1 A_2 \phi_{\alpha, a_2} = a_2 A_1 \phi_{\alpha, a_2} = a_2 \cdot A_1 \cdot \frac{1}{C} \psi_{\alpha, a_1} = a_2 a_1 \cdot \frac{1}{C} \psi_{\alpha, a_1} = a_2 a_1 \phi_{\alpha, a_2}$$

$$A_2 A_1 \phi_{\alpha, a_2} = \frac{1}{C} A_2 A_1 \psi_{\alpha, a_1} = \frac{a_1}{C} A_2 \psi_{\alpha, a_1} = a_1 A_2 \phi_{\alpha, a_2} = a_1 a_2 \phi_{\alpha, a_2}$$

$$\Rightarrow A_1 A_2 \phi_{\alpha, a_2} = A_2 A_1 \phi_{\alpha, a_2} \text{ for all } \alpha \Rightarrow A_1, A_2 \text{ are compatible} \quad \times$$

\therefore Energy eigenstates are degenerate \times

5. /D

$$[\theta_x L_x + \theta_y L_y + \theta_z L_z, x^2 + y^2 + z^2]$$

$$= \left[\sum_{i=1}^3 \theta_i \epsilon_{ijk} r_j P_k, \sum_{\ell=1}^3 r_\ell^2 \right]$$

$$= \sum_{i,\ell=1}^3 [\theta_i \epsilon_{ijk} r_j P_k, r_\ell^2]$$

$$= \sum_{i,\ell=1}^3 \theta_i \epsilon_{ijk} (r_\ell [r_j P_k, r_\ell] + [r_j P_k, r_\ell] r_\ell)$$

$$= \sum_{i,\ell=1}^3 \theta_i \epsilon_{ijk} (r_\ell \cdot (-ik) \delta_{k\ell} \cdot r_j + r_j \cdot (-ik) \delta_{k\ell} r_\ell)$$

$$= \sum_{i=1}^3 \theta_i \epsilon_{ijk} (-ik) \cdot 2 \cdot r_j r_k = 0$$

$$\therefore \left[\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}, x^2 + y^2 + z^2 \right] = 0$$

$$\Rightarrow \exp \left[\frac{i}{\hbar} \vec{\theta} \cdot \vec{L} \right] (x^2 + y^2 + z^2) \exp \left[- \frac{i}{\hbar} \vec{\theta} \cdot \vec{L} \right]$$

$$= x^2 + y^2 + z^2 + \left[\frac{i}{\hbar} \vec{\theta} \cdot \vec{L} \right] \cancel{x^2 + y^2 + z^2} + \dots$$

$$= x^2 + y^2 + z^2 \quad \text{※}$$

$$3. \quad [\mathbf{Q}, \mathbf{H}] = \left[\mathbf{Q}, \frac{\mathbf{P}^2}{2m} + V(\mathbf{Q}) \right] = \frac{ik}{m} \mathbf{P}$$

$$\Rightarrow [\mathbf{Q}, [\mathbf{Q}, \mathbf{H}]] = \left[\mathbf{Q}, \frac{ik}{m} \mathbf{P} \right] = \frac{ik}{m} \cdot ik \mathbb{1} = -\frac{k^2}{m} \mathbb{1}$$

$$\therefore (\psi_s, [\mathbf{Q}, [\mathbf{Q}, \mathbf{H}]] \psi_s) = -\frac{k^2}{m} (\psi_s, \mathbb{1} \psi_s) = -\frac{k^2}{m}$$

$$\text{Because } [Q, [Q, H]] = [Q, QH - HQ] \\ = QQH - 2QHQ + HQQ$$

$$\text{Let } Q\psi_s = \sum_k a_k \psi_k$$

$$\Rightarrow (\psi_\ell, Q\psi_s) = \sum_k a_k (\psi_\ell, \psi_k) = a_\ell \Rightarrow a_\ell = Q_{\ell s}$$

$$\begin{aligned} \therefore (\psi_s, QQ\psi_s) &= (Q^+ \psi_s, Q\psi_s) = \sum_{\ell, k} Q_{\ell s}^* Q_{ks} (\psi_\ell, \psi_k) \\ &= \sum_{\ell, k} Q_{\ell s}^* Q_{ks} \delta_{\ell k} = \sum_k |Q_{ks}|^2 \end{aligned}$$

$$\Rightarrow (\psi_s, QHQ\psi_s) = E_s (\psi_s, QQ\psi_s) = E_s \sum_k |Q_{ks}|^2$$

$$\begin{aligned} (\psi_s, QHQ\psi_s) &= (Q^+ \psi_s, HQ\psi_s) = \sum_\ell \sum_k Q_{\ell s}^* Q_{ks} (\psi_\ell, H\psi_k) \\ &= \sum_\ell \sum_k Q_{\ell s}^* Q_{ks} E_k \delta_{\ell k} = \sum_k E_k |Q_{ks}|^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow -\frac{\hbar^2}{m} &= (\psi_s, [Q, [Q, H]] \psi_s) \\ &= (\psi_s, (QQH - 2QHQ + HQQ), \psi_s) \\ &= \sum_k |Q_{ks}|^2 (E_s - E_k) \end{aligned}$$

$$\Rightarrow \frac{\hbar^2}{2m} = \sum_k (E_k - E_s) |Q_{ks}|^2$$