Lorentz Symmetry, Weyl Spinors, Chirality and Dirac Equation

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Minkowski Space and Lorentz Transformation

Difine the Minkowski contravariant 4-vector as:

$$x^\mu = (x^0 = ct, \vec{x}), \tag{1}$$

and the Minkowski covariant 4-vector as:

$$x_\mu = (x^0 = -ct, \vec{x}), \tag{2}$$

with the metric tensor

$$g_{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu \ ; \qquad -g_{00} = g_{11} = g_{22} = g_{33} = 1. \tag{3}$$

A linear transformation on the x^{μ} given as follows

$$x'^{\mu} = \mathbf{\Lambda}^{\mu}_{\nu} \tag{4}$$

is called Homogeneous Lorentz transformation (HLT), or simply LT if the following condition is met:

$$x'^{\mu} x'_{\mu} = x^{\mu} \tag{5}$$

or in matrix notation as

$$\mathbf{\Lambda}^{T} \mathbf{g} \mathbf{\Lambda} = \mathbf{g} \quad \text{or} \quad \mathbf{\Lambda}^{T} \mathbf{g} = \mathbf{g} \mathbf{\Lambda}^{-1}. \tag{6}$$

Since there exists an identity Lorentz transformation,$\Lambda = \mathbf{I}$, and an inverse Lorentz transformation, $\mathbf{\Lambda}^{-1}$, namely both $\mathbf{I}$ and $\mathbf{\Lambda}^{-1}$ exist.
Therefore LT forms a group SO(3,1) because:

$$(\mathbf{\Lambda}_1\mathbf{\Lambda}_2)^T g(\mathbf{\Lambda}_1\mathbf{\Lambda}_2) = \mathbf{\Lambda}_2^T\mathbf{\Lambda}_1^T g\mathbf{\Lambda}_1\mathbf{\Lambda}_2 = g. \qquad (7)$$

The condition $\det\mathbf{\Lambda} = 1$ is automatically satisfied. We shall only consider the proper LT in which $\mathbf{\Lambda}_0^0 \geqslant 1$ in this lecture.

Since the condition $\mathbf{\Lambda}^t g \mathbf{\Lambda} = g$ provide 10 constraints among 16 matrix elements of $\boldsymbol{\lambda}$, the remaining 6 independent coefficients serve as the 6 group parameters, specified as $\mathbf{\Lambda} = \mathbf{\Lambda}(\vec{\theta}, \vec{\xi})$ and

$$\vec{\theta} = (\theta^1, \theta^2, \theta^3) = \text{ rotation,} \tag{8a}$$

$$\vec{\xi} = (\xi^1, \xi^2, \xi^3) = \text{ Lorentz boost.} \tag{8b}$$

Generators of Lorentz Group

The generators of the group are given:

$$A_i = \frac{\partial}{\partial \theta^i} \mathbf{\Lambda}(\vec{\theta}, \vec{\xi}), \quad B_i = \frac{\partial}{\partial \xi^i} \mathbf{\Lambda}(\vec{\theta}, \vec{\xi}), \tag{8a,b}$$

For the Lorentz boost along 1-axis with angle ξ,

$$\mathbf{\Lambda} = \begin{pmatrix} \cosh \xi & -\sinh \xi & 0 & 0 \\ -\sinh \xi & \cosh \xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

where $\xi = \tanh^{-1} \beta$ and

$$B_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10a}$$

similarly,

$$B_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{10b,c}$$

And for the generators if rotation, we have

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{11a}$$

and

$$\text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{11b,c}$$

SO(3,1) Lie algebra as:

$$[A_i, A_j] = -\epsilon_{ij}^k A_k, \quad [A_i, B_j] = -\epsilon_{ij}^k B_k, \tag{12}$$

Canonical formulation of algebra:

$$M^{\mu\nu} = x^\mu \frac{\partial}{\partial x_\nu} - x^\nu \frac{\partial}{\partial x_\mu}, \tag{13a}$$

$$[M^{\mu\nu}, M^{\alpha\beta}] = -g^{\nu\beta} M^{\mu\alpha} - g^{\mu\alpha} M^{\nu\beta} + g^{\nu\alpha} M^{\mu\beta} + g^{\mu\beta} M^{\nu\alpha}. \tag{13b}$$

If we denote

$$L_i = \frac{1}{2}\left(\frac{A_i}{i} + B_i\right), \quad \text{and} \quad R_i = \frac{1}{2}\left(\frac{A_i}{i} - B_i\right). \quad \text{(14a,b)}$$

The algebra takes as

$$[L_i, L_j] = i\epsilon_{ij}^{k} L_k, \quad \text{(15a)}$$

$$[L_i, R_i] = 0, \quad \text{(15b)}$$

$$[R_i, R_j] = i\epsilon_{ij}^{k} R_k. \quad \text{(15c)}$$

Irreducible Representations of Lorentz Group and Weyl Spinors

Consider the finite dimensional representations, denoted by (l, r) with the basis

$$|l, m\rangle \otimes |r, n\rangle \equiv |l, m; r, n\rangle \qquad (16)$$

where

$$-l \leqslant m \leqslant l, \quad -r \leqslant n \leqslant r, \quad \text{and} \quad l, r = \text{ half integers.} \qquad (17)$$

The simpliest representation of the generators, an one dimensional $(0, 0)$-representation read as

$$\langle 0, 0; 0, 0|L_i|0, 0; 0, 0\rangle = \langle 0, 0; 0, 0|R_i|0, 0; 0, 0\rangle = 0, \qquad (18)$$

$(\frac{1}{2}, 0)$-representation: left-handed-spinor

$$|\frac{1}{2}, m; 0, 0\rangle \tag{19}$$

$(0, \frac{1}{2})$-representation: right-handed-spinor

$$|0, 0; \frac{1}{2}, n\rangle \tag{20}$$

then we have

$$L_i^{(\frac{1}{2},0)} = \frac{1}{2}\boldsymbol{\sigma}_{\mathbf{i}}, \qquad R_i^{(\frac{1}{2},0)} = 0, \tag{0.21a}$$

$$L_i^{(0,\frac{1}{2})} = 0, \qquad R_i^{(0,\frac{1}{2})} = \frac{1}{2}\boldsymbol{\sigma}_{\mathbf{i}}, \tag{0.21b}$$

which lead to

$$A_i^{(\frac{1}{2},0)} = \frac{i}{2}\boldsymbol{\sigma}_{\mathbf{i}}, \qquad B_i^{(\frac{1}{2},0)} = \frac{1}{2}\boldsymbol{\sigma}_{\mathbf{i}}, \tag{0.22a}$$

$$A_i^{(0,\frac{1}{2})} = \frac{i}{2}\boldsymbol{\sigma}_{\mathbf{i}}, \qquad B_i^{(0,\frac{1}{2})} = -\frac{1}{2}\boldsymbol{\sigma}_{\mathbf{i}}, \tag{0.22b}$$

and the 2-dimensional irreducible representation of Lorentz group as

$$D^{(\frac{1}{2},0)}(\vec{\theta},\vec{\xi}) = \exp\left(\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})\right), \tag{23a}$$

$$D^{(0,\frac{1}{2})}(\vec{\theta},\vec{\xi}) = \exp\left(\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})\right). \tag{23b}$$

As a quick check that

$$D^{(\frac{1}{2},0)\dagger}(\vec{\theta},\vec{\xi}) = \exp\left(-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})\right) \neq D^{(\frac{1}{2},0)}(\vec{\theta},\vec{\xi})^{-1},$$

$$D^{(0,\frac{1}{2})\dagger}(\vec{\theta},\vec{\xi}) = \exp\left(-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})\right) \neq D^{(0,\frac{1}{2})}(\vec{\theta},\vec{\xi})^{-1},$$

Let us perform the identifications

$$|\frac{1}{2},\frac{1}{2}\rangle_l \longmapsto \begin{pmatrix}1\\0\end{pmatrix} = e_1, \quad |\frac{1}{2},-\frac{1}{2}\rangle_l \longmapsto \begin{pmatrix}0\\1\end{pmatrix} = e_2, \quad (25a,b)$$

then

$$\psi_l(x) = \psi_l^a(x)e_a = \begin{pmatrix} \psi_l^1(x) \\ \psi_l^2(x) \end{pmatrix}, \tag{26}$$

the Lorentz Transformation as

$$\psi_l(x) \longmapsto \psi_l'(x') = D^{(\frac{1}{2},0)}(\vec{\theta},\vec{\xi})\psi_l(\mathbf{\Lambda}^{-1}x'), \tag{27}$$

Similarly if

$$\psi_r(x) = \psi_r^a(x)f_a = \psi_r^1(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_r^2(x)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \psi_r^1(x) \\ \psi_r^2(x) \end{pmatrix}, \tag{28}$$

the transformation reads as

$$\psi_r(x) \longmapsto \psi_r'(x') = D^{(0,\frac{1}{2})}(\vec{\theta},\vec{\xi})\psi_r(\mathbf{\Lambda}^{-1}x')). \tag{29}$$

SO(3,1) and SL(2,C)

SL(2,C) transformation in $\mathcal{C}^2$-space:

$$\xi' = \begin{pmatrix} \xi'^1 \\ \xi'^2 \end{pmatrix} = \mathbf{L} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}. \tag{30}$$

where

$$\det \mathbf{L} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ab - bc = 1, \tag{31}$$

If we exponentiate L by a 2×2 matrix $\mathbf{A}$, i.e.

$$\mathbf{L} = e^{\mathbf{A}}, \tag{32}$$

Then we have the following proposition

Proposition 1.

If a matrix $\mathbf{L}$ *can be expressed as* $\mathbf{L} = e^{\mathbf{A}}$, *then*

$$\det \mathbf{L} = e^{\mathrm{Tr}\ \mathbf{A}}. \tag{33}$$

Hence we ensure that

$$\det D^{(\frac{1}{2},0)}(\vec{\theta},\vec{\xi}) = \det e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})} = e^{\mathrm{Tr}\ \frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})} = 1, \tag{34}$$

and

$$\det D^{(0,\frac{1}{2})}(\vec{\theta},\vec{\xi}) = \det e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})} = e^{\mathrm{Tr}\ \frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})} = 1. \tag{35}$$

The isomorphism of SL(2,C) onto SO(3,1) in Lorentz transformation can be demonstrated as follows:
let

$$\mathbf{X} = x^\mu \boldsymbol{\sigma}_\mu = \begin{pmatrix} -x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^0 - x^3 \end{pmatrix}, \tag{36}$$

and

$$\det \mathbf{X} = (x^0)^2 - \vec{x}^2. \tag{37}$$

which leads to the Lorentz Transformation on **X** as

$$\mathbf{X}' = D^{(\frac{1}{2},0)}(\vec{\theta}, \vec{\xi})\mathbf{X}D^{(\frac{1}{2},0)\dagger}(\vec{\theta}, \vec{\xi}), \tag{38}$$

because of the invariance of the length of the space-time vector,

$$\det \mathbf{X}' = \det \mathbf{X}. \tag{39}$$

As an example when $\mathcal{O}'$-frame is boost along the 3rd axis, i.e.

$$\mathbf{X}' = e^{\frac{1}{2}\boldsymbol{\sigma}_3\xi}\mathbf{X}e^{\frac{1}{2}\boldsymbol{\sigma}_3\xi} = \begin{pmatrix} e^{\frac{1}{2}\xi} & 0 \\ 0 & e^{-\frac{1}{2}\xi} \end{pmatrix} \mathbf{X} \begin{pmatrix} e^{\frac{1}{2}\xi} & 0 \\ 0 & e^{-\frac{1}{2}\xi} \end{pmatrix}. \tag{40}$$

we regain the LT as follows

$$x'^0 = \cosh\xi x^0 - \sinh\xi x^3, \tag{41a}$$
$$x'^1 = x^1, \tag{41b}$$
$$x'^2 = x^2, \tag{41c}$$
$$x'^3 = -\sinh\xi x^0 + \cosh\xi x^3. \tag{41d}$$

Chiral Transformation

$\mathcal{K}$ = chiral operator, which is a discrete transformation between left handed irreducible representations and the right handed irreducible representations
It is an antilinear operator, i.e.

$$\mathcal{K}(a\psi + b\varphi) = a^*\mathcal{K}\psi + b^*\mathcal{K}\varphi, \tag{42}$$

as well as an antiunitary operator:

$$(\mathcal{K}\psi, \mathcal{K}\varphi) = (\varphi, \psi) = (\psi, \varphi)^*. \tag{43}$$

It is nothing to with the space-time coordinates, hence

$$\mathcal{K}A_i\mathcal{K}^{-1} = A_i, \quad \mathcal{K}B_i\mathcal{K}^{-1} = B_i. \tag{44}$$

but the operators L_i and R_i transform as follows

$$\mathcal{K}L_i\mathcal{K}^{-1} = \frac{1}{2}\mathcal{K}\left(\frac{A_i}{i} + B_i\right)\mathcal{K}^{-1} = -R_i, \quad \mathcal{K}R_i\mathcal{K}^{-1} = -L_i, \tag{45a,b}$$

therefore we reach the following proposition if the basis of $(\frac{1}{2}, 0)$- and $(0, \frac{1}{2})$-representation are abbreviated by

$$|j, m; 0, 0\rangle = L_{jm}, \tag{46a}$$
$$|0, 0, k, n\rangle = R_{kn}, \tag{46b}$$

then

Proposition 2.

The vector $\mathcal{K}L_{jm}$ is the eigenvector of R^2 and R_3 with the eigenvalues $j(j+1)$ and $-m$ respectively. While the vector $\mathcal{K}R_{kn}$ is the eigenvector of L^2 and L_3 with the eigenvalues $k(k+1)$ and $-n$ respectively.

Since $^2 = R^2\mathcal{K}$, $\mathcal{K}\vec{L} = -\vec{R}\mathcal{K}$

then we have

$$R^2\mathcal{K}L_{jm} = \mathcal{K}L^2L_{jm} = j(j+1)\mathcal{K}L_{jm},$$
$$R_3\mathcal{K}L_{jm} = -\mathcal{K}L_3L_{jm} = -m\mathcal{K}L_{jm},$$

Therefore

$$\mathcal{K}L_{jm} = \gamma(m)R_{j-m}.$$

Similarly

$$\mathcal{K}R_{kn} = \delta nL_{k-n}.$$

Hence we have

$$(\mathcal{K}L_{jm}, \mathcal{K}L_{jm'}) = \gamma^*(m)\gamma(m')(R_{j,-m}, R_{j,-m'}) = (L_{jm'}, L_{jm}), \tag{47}$$

or $\gamma^*(m)\gamma(m')\delta_{-m,-m'} = \delta_{m'm}$

Take spinor for instance, we obtain that

$$\mathcal{K}L_{\frac{1}{2}m} = \gamma(n)\mathcal{K}R_{\frac{1}{2}-m}, \qquad \mathcal{K}R_{\frac{1}{2}n} = \delta(n)\mathcal{K}L_{\frac{1}{2}-n}, \quad \text{(48ab)}$$

or

$$\mathcal{K}\begin{pmatrix}1\\0\end{pmatrix}_l = \gamma(\frac{1}{2})\begin{pmatrix}0\\-1\end{pmatrix}_r \qquad \mathcal{K}\begin{pmatrix}0\\1\end{pmatrix}_l = \gamma(-\frac{1}{2})\begin{pmatrix}-1\\0\end{pmatrix}_r, \quad \text{(49a,b)}$$

$$\mathcal{K}\begin{pmatrix}1\\0\end{pmatrix}_r = \delta(\frac{1}{2})\begin{pmatrix}0\\-1\end{pmatrix}_l \qquad \mathcal{K}\begin{pmatrix}0\\1\end{pmatrix}_r = \delta(-\frac{1}{2})\begin{pmatrix}-1\\0\end{pmatrix}_l, \quad \text{(49c,d)}$$

Let us evaluate the following matrix elements

$$
\begin{aligned}
D^{(\frac{1}{2},0)}_{mm'} &= (L_{\frac{1}{2},m}, e^{\vec{\theta}\cdot\vec{A}+\vec{\xi}\cdot\vec{B}} L_{\frac{1}{2},m'}) = (\mathcal{K} e^{\vec{\theta}\cdot\vec{A}+\vec{\xi}\cdot\vec{B}} L_{\frac{1}{2},m'}, \mathcal{K} L_{\frac{1}{2},m}) \\
&= \gamma^*(m')\gamma(m)(e^{\vec{\theta}\cdot\vec{A}+\vec{\xi}\cdot\vec{B}} R_{\frac{1}{2},-m'}, R_{\frac{1}{2},-m}) \\
&= \gamma(m) D^{(0,\frac{1}{2})*}_{-m,-m'} \gamma^*(m'),
\end{aligned}
\tag{50}
$$

or

$$
D^{(\frac{1}{2},0)} = \begin{pmatrix} 0 & \gamma(\frac{1}{2}) \\ \gamma(-\frac{1}{2}) & 0 \end{pmatrix} D^{(0,\frac{1}{2})*} \begin{pmatrix} 0 & \gamma^*(-\frac{1}{2}) \\ \gamma^*(\frac{1}{2}) & 0 \end{pmatrix}. \tag{51}
$$

Similarly,

$$D^{(0,\frac{1}{2})} = \begin{pmatrix} 0 & \delta(\frac{1}{2}) \\ \delta(-\frac{1}{2}) & 0 \end{pmatrix} D^{(\frac{1}{2},0)*} \begin{pmatrix} 0 & \delta^*(-\frac{1}{2}) \\ \delta^*(\frac{1}{2}) & 0 \end{pmatrix}. \quad (52)$$

In order to be consistent with the LT for spinors, one chooses

$$\gamma(\frac{1}{2}) = \delta(\frac{1}{2}) = -\gamma(-\frac{1}{2}) = -\delta(-\frac{1}{2}) = 1 \quad (53)$$

hence we have

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (54)$$

and

$$\epsilon \boldsymbol{\sigma}_i^* \epsilon^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{\sigma}_i^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\boldsymbol{\sigma}_i. \quad (55)$$

Spinor space and Co-spinor space

$$\text{Let}\ \begin{cases} \mathcal{V}_{(\frac{1}{2},0)} & : \ \text{left-hand spinor space} \\ e_a & : (a=1,2)\ \text{two left-handed spinor} \end{cases} \tag{56}$$

A spinor in $\mathcal{V}_{(\dot{\frac{1}{2}},0)}$

$$\begin{cases} \mathcal{V}_{(\dot{\frac{1}{2}},0)} & : \ \text{co-left-hand spinor space} \\ e_{\dot{a}} & : (a=1,2)\ \text{two co-left-handed spinor} \end{cases} \tag{57}$$

which is related to the left-handed spinor by

$$\dot{\psi} = \psi^T \epsilon^T = (\dot{\psi}_1, \dot{\psi}_1) = (\psi^1, \psi^2)\epsilon^T = (\psi^1, \psi^2)\epsilon^{-1}. \tag{58}$$

and the corresponding LT is given as

$$\dot{\psi} \overset{L.T.}{\longmapsto} \dot{\psi}' = \psi'^T \epsilon^{-1} = \psi^T D^{T(\frac{1}{2},0)} \epsilon^{-1} \tag{59}$$

$$= \dot{\psi}(\epsilon D^{(\frac{1}{2},0)*} \epsilon^{-1})^\dagger = \dot{\psi} D^{(0,\frac{1}{2})\dagger}. \tag{60}$$

Similarly the LT for the co-right-handed spinor is given as

$$\dot{\varphi} \overset{L.T.}{\longmapsto} \dot{\varphi}' = \varphi'^T \epsilon^T = \dot{\varphi} D^{(\frac{1}{2},0)\dagger}. \tag{61}$$

Let us construct the 4-dimensional product space as

$$\mathcal{V}_{(\frac{1}{2},\dot{\frac{1}{2}})} : \ e_a \dot{f}^b \ \text{ as basis} \tag{62a}$$

$$\mathcal{V}_{(\dot{\frac{1}{2}},\frac{1}{2})} : \ \dot{e}_a f^b \ \text{ as basis} \tag{62b}$$

then any element in $\mathcal{V}_{(\frac{1}{2},\frac{\dot{1}}{2})}$ and in $\mathcal{V}_{(\frac{\dot{1}}{2},\frac{1}{2})}$ can be written respectively as

$$U^{(\frac{1}{2},\frac{\dot{1}}{2})} = \begin{pmatrix} u^1_{\dot{1}} & u^1_{\dot{2}} \\ u^2_{\dot{1}} & u^2_{\dot{2}} \end{pmatrix}. \tag{63}$$

and

$$U^{(\frac{\dot{1}}{2},\frac{1}{2})} = \begin{pmatrix} u^{\dot{1}}_1 & u^{\dot{2}}_1 \\ u^{\dot{1}}_2 & u^{\dot{2}}_2 \end{pmatrix}. \tag{64}$$

It is obvious that the space-time matrix

$$\mathbf{X} = x^\mu \boldsymbol{\sigma}_\mu = \begin{pmatrix} -x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^0 - x^3 \end{pmatrix}, \tag{65}$$

transforms as an element of $\mathcal{V}_{(\frac{1}{2},\frac{\dot{1}}{2})}$-representation, while

$$\mathbf{X}_c = \epsilon \mathbf{X}^* \epsilon^{-1} \tag{66}$$

We are now in the position to emphasize the next proposition

Proposition 3.

An operator of the $(\frac{1}{2},\frac{\dot{1}}{2})$*-representation acts upon a vector of the* $(0,\frac{1}{2})$*-representation yields a vector of* $(\frac{1}{2},0)$*-representation. Conversely an operator of the* $(\frac{\dot{1}}{2},\frac{1}{2})$*-representation acts upon a vector of the* $(\frac{1}{2},0)$*-representation will yield a vector of the* $(0,\frac{1}{2})$*-representation.*

The proof goes as; If $\mathbf{A} \in (\frac{1}{2},\frac{\dot{1}}{2})$-representation, $\xi \in (0,\frac{1}{2})$-representation, then

$$\eta \xmapsto{L.T.} \eta' = \mathbf{A}'\xi' = D^{(\frac{1}{2},0)}\mathbf{A}D^{(\frac{1}{2},0)\dagger}D^{(0,\frac{1}{2})}\xi = D^{(\frac{1}{2},0)}\eta, \quad (67)$$

Hence we have a left-handed spinor, i.e.

$$\eta = \mathbf{A}\xi, \tag{68}$$

Similarly that

$$\text{if} \begin{cases} \mathbf{A} & \in (\frac{1}{2}, \frac{1}{2}) - \text{representation}, \\ \eta & \in (\frac{1}{2}, 0) - \text{representation}. \end{cases}$$

Therefore we reach as follows,

$$\xi \stackrel{L.T.}{\longmapsto} \xi' = \mathbf{A}'_c \eta' = D^{(0,\frac{1}{2})} \mathbf{A}_c D^{(0,\frac{1}{2})\dagger} D^{(\frac{1}{2},0)} \xi = D^{(0,\frac{1}{2})} \xi, \tag{69}$$

Dirac spinor and Dirac equation

Since any Lorentz 4-vector with the construction

$$U = \sigma_\mu u^\mu, \qquad U_c = \sigma^c_\mu u^\mu, \tag{70}$$

transforms as $(\frac{1}{2}, \frac{\dot{1}}{2})$-representation and $(\frac{\dot{1}}{2}, \frac{1}{2})$-representation respectively. Therefore,

$$\mathbf{P}_c \psi_l = m_0 c \psi_r, \tag{0.71a}$$
$$\mathbf{P} \psi_r = m_0 c \psi_l, \tag{0.71b}$$

hence we have

$$\begin{pmatrix} 0 & \mathbf{P}_c \\ \mathbf{P} & 0 \end{pmatrix} \begin{pmatrix} \psi_r \\ \psi_l \end{pmatrix} = m_0 c \begin{pmatrix} \psi_r \\ \psi_l \end{pmatrix}, \tag{72}$$

If we define the Dirac spinor ψ_d as $\psi_r \oplus \psi_l$, then

$$\begin{pmatrix} 0 & \mathbf{P}_c \\ \mathbf{P} & 0 \end{pmatrix} \psi_d(x) = m_0 c \psi_d(x), \tag{73}$$

or

$$\left[\begin{pmatrix} 0 & i\boldsymbol{\sigma}_c^{\mu} \\ i\boldsymbol{\sigma}^{\mu} & 0 \end{pmatrix} \partial_\mu + \frac{m_0 c}{\hbar} \right] \psi_d(x) = 0, \tag{74}$$

which can be cast into

$$\left(i\gamma^{\mu}\partial_\mu + \frac{m_0 c}{\hbar} \right) \psi_d(x) = 0, \tag{75}$$

with γ^μ defined as

$$\gamma^\mu = \begin{pmatrix} 0 & \boldsymbol{\sigma}_c^\mu \\ \boldsymbol{\sigma}^\mu & 0 \end{pmatrix} \quad \text{or} \quad \gamma^0 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\boldsymbol{\sigma}^i \\ \boldsymbol{\sigma}^i & 0 \end{pmatrix}. \tag{76}$$

The covariant formulation of Dirac equation does not imply that

$$\gamma_\mu \partial^\mu = \gamma'_\mu \partial'^\mu$$

In fact that we have the following proposition.

Proposition 4.

The gamma matrices γ^μ are universal in all Lorentz frame, namely the Dirac equation in another Lorentz frame, i.e. the $\mathcal{O}'$-system always takes the same gamma matrices γ^μ used in $\mathcal{O}$-system. The equation in $\mathcal{O}'$-system is expressed as

$$\left(i\gamma^\mu \partial'_\mu + \frac{m_0 c}{\hbar}\right)\psi'_d(x') = 0.$$

It can be proved by defining

$$D(\vec{\theta},\vec{\xi}) = D^{(0,\frac{1}{2})}(\vec{\theta},\vec{\xi}) \oplus D^{(\frac{1}{2},0)}(\vec{\theta},\vec{\xi})$$

and multiplying it upon the Dirac equation as follows

$$\left(iD\gamma^\mu D^{-1}\partial_\mu + \frac{m_0 c}{\hbar}\right) D\psi_d(x) = 0. \tag{77}$$

One can show that

$$D\gamma^\mu D^{-1} = \mathbf{\Lambda}^\mu_\nu \gamma^\nu, \tag{78}$$

and the Dirac equation in the new Lorentz frame, i.e. $\mathcal{O}'$frame reads as

$$\left(i\gamma^\mu \partial'_\mu + \frac{m_0 c}{\hbar}\right) \psi'_d(x') = 0, \tag{79}$$

by identifying $\psi'_d(x') \equiv D(\vec{\theta},\vec{\xi})\psi_d(x) = D(\vec{\theta},\vec{\xi})\psi_d(\mathbf{\Lambda}^{-1}x')$.

Zero Mass Limit and Helicity of Weyl spinors

Last demonstration for zero mass limit in Dirac equation

$$\begin{pmatrix} 0 & \mathbf{P}_c \\ \mathbf{P} & 0 \end{pmatrix} \psi_d(x) - m_0 c \psi_d(x) = 0. \tag{80}$$

For the limit that $m = 0$, then

$$\mathbf{P}_c \psi_l = 0, \quad \mathbf{P} \psi_r = 0. \tag{81}$$

and $p^0 = |\vec{p}|$, which implies that

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}) \psi_r = \psi_r, \quad (\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}) \psi_l = -\psi_l, \tag{82}$$