

Problem Set 1.

B99602056 莊道茂

Ex 1 [10]

Take $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$, $x \in [-1, 1]$.

By applying Gram-Schmidt process, we obtain that

$$\Rightarrow \psi_1 = \frac{|f_1\rangle}{\sqrt{\langle f_1 | f_1 \rangle}} = \frac{x}{\sqrt{\int_{-1}^1 x^2 dx}} = \sqrt{\frac{3}{2}} x$$

by using bracket notation.

$$\begin{aligned} \psi'_2 &= f_2(x) - |\psi_1\rangle \langle \psi_1 | f_2 \rangle = x^2 - \sqrt{\frac{3}{2}} x \left(\int_{-1}^1 \sqrt{\frac{3}{2}} x \cdot x^2 dx \right) \\ &= x^2 - \left(\sqrt{\frac{3}{2}} \right)^2 x \underbrace{\left[\frac{1}{4} x^4 \Big|_{-1}^1 \right]}_{\text{odd func.}} \\ &= x^2 \end{aligned}$$

$$\Rightarrow \psi_2 = \frac{|\psi'_2\rangle}{\sqrt{\langle \psi'_2 | \psi'_2 \rangle}} = \frac{x^2}{\sqrt{\int_{-1}^1 x^4 dx}} = \sqrt{\frac{5}{2}} x^2$$

$$\begin{aligned} \psi'_3 &= f_3(x) - |\psi_1\rangle \langle \psi_1 | f_3 \rangle - |\psi_2\rangle \langle \psi_2 | f_3 \rangle \\ &= x^3 - \sqrt{\frac{3}{2}} x \left(\int_{-1}^1 \sqrt{\frac{3}{2}} x \cdot x^3 dx \right) - \sqrt{\frac{5}{2}} x^2 \left(\int_{-1}^1 \sqrt{\frac{5}{2}} x^2 \cdot x^3 dx \right) \\ &= x^3 - \sqrt{\frac{3}{2}} x \left(\sqrt{\frac{3}{2}} \cdot \frac{1}{5} (1+1) \right) - \sqrt{\frac{5}{2}} x^2 \sqrt{\frac{5}{2}} \underbrace{\left(\frac{1}{6} x^6 \Big|_{-1}^1 \right)}_{\text{odd}} \\ &= x^3 - \frac{3}{2} \cdot \frac{2}{5} \cdot x \\ &= x^3 - \frac{3}{5} x \end{aligned}$$

$$\begin{aligned} \Rightarrow \psi_3 &= \frac{|\psi'_3\rangle}{\sqrt{\langle \psi'_3 | \psi'_3 \rangle}} = \frac{\left(x^3 - \frac{3}{5} x \right)}{\sqrt{\int_{-1}^1 \left(x^3 - \frac{3}{5} x \right)^2 dx}} = \frac{x^3 - \frac{3}{5} x}{\sqrt{\int_{-1}^1 \left(x^6 - \frac{6}{5} x^4 + \frac{9}{25} x^2 \right) dx}} \\ &= \frac{\sqrt{175}}{8} \left(x^3 - \frac{3}{5} x \right) \\ &= \frac{5}{2} \sqrt{\frac{7}{2}} \left(x^3 - \frac{3}{5} x \right) \end{aligned}$$

Therefore, the first 3 orthonormal functions are

$\psi_1(x) = \sqrt{\frac{3}{2}} x$, $\psi_2(x) = \sqrt{\frac{5}{2}} x^2$, $\psi_3(x) = \frac{5}{2} \sqrt{\frac{7}{2}} \left(x^3 - \frac{3}{5} x \right)$, which is like a normal Legendre-polynomial.

Ex. 2: $\| \psi + \phi \|_2^2 = (\psi + \phi, \psi + \phi)$ (in Hilbert-space, with norm: $\| \psi \| = \sqrt{(\psi, \psi)}$)

(1.3a) $\stackrel{=} {(\psi + \phi, \psi) + (\psi + \phi, \phi)}$ $\stackrel{\text{in bracket notation}}{=} \sqrt{(\psi, \psi)}$

(1.3c) $\stackrel{=} {(\psi, \psi + \phi)^* + (\phi, \psi + \phi)^*}$

(1.3a)
again $\stackrel{=} {[(\psi, \psi) + (\psi, \phi)]^* + [(\phi, \psi) + (\phi, \phi)]^*}$
 $= (\psi, \psi)^* + (\psi, \phi)^* + (\phi, \psi)^* + (\phi, \phi)^*$
 $= (\psi, \psi) + (\phi, \psi) + (\psi, \phi) + (\phi, \phi)$
 $= \| \psi \|^2 + \| \phi \|^2 + (\phi, \psi) + (\psi, \phi)$
 $= \| \psi \|^2 + \| \phi \|^2 + [(\phi, \psi) + (\phi, \psi)^*]$
 $= \| \psi \|^2 + \| \phi \|^2 + 2 \operatorname{Re} \{ (\phi, \psi) \}$
 \uparrow
real part. notation.

(since $|(\psi, \phi)| \geq \operatorname{Re} \{ (\psi, \phi) \}$)

$\leq \| \psi \|^2 + \| \phi \|^2 + 2 |(\psi, \phi)|$

$\leq \underbrace{\| \psi \|^2 + \| \phi \|^2}_{\| (\psi + \phi) \|^2} + 2 \| \psi \| \| \phi \|$

Finally, $\Rightarrow \boxed{\| \psi + \phi \|_2^2 \leq (\| \psi \| + \| \phi \|)^2}$

Ex. 3. 10

By Ex. 2. we derived the following relation $\textcircled{1}$.

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + (x, y) + (y, x) \quad \textcircled{1}$$

thus, if we take y into $-y$, then we can obtain:

$$\|x+(-y)\|^2 = \|x\|^2 + \|(-y)\|^2 + (x, -y) + (-y, x)$$

$$\Rightarrow \|x-y\|^2 = \|x\|^2 + \|y\|^2 - (x, y) - (y, x) \quad \textcircled{2}$$

By (1.3b) in lecture notes.

Now, substituting ψ, ϕ into x and y in equation $\textcircled{1}$ and $\textcircled{2}$.

$$\Rightarrow \|\psi+\phi\|^2 = \|\psi\|^2 + \|\phi\|^2 + (\psi, \phi) + (\phi, \psi) \quad \textcircled{3}$$

$$\Rightarrow \begin{cases} \|\psi+\phi\|^2 = \|\psi\|^2 + \|\phi\|^2 + (\psi, \phi) + (\phi, \psi) \\ \|\psi-\phi\|^2 = \|\psi\|^2 + \|\phi\|^2 - (\psi, \phi) - (\phi, \psi) \end{cases} \quad \textcircled{4}$$

$\textcircled{3}$ plus $\textcircled{4}$

$$\Rightarrow \|\psi+\phi\|^2 + \|\psi-\phi\|^2 = 2(\|\psi\|^2 + \|\phi\|^2)$$

Ex. 4 10.

(Proof 1)

(i) If $(V, \|\cdot\|)$ is a normed vector space, the norm $\|\cdot\|$ induces a metric, thus there is a topological structure on V .

This metric is defined in the natural way, the distance between two vectors u, v is given by $\|u - v\|$.

(ii) Since all norms on a finite-dimensional vector space are equivalent from topological point of view (I will try to briefly prove this property!) although the resulting metric spaces need not be the same.

(iii) And, since any Euclidean space is complete (I take this property for granted, since every Cauchy sequences in it are converge!!) Thus, it follows that all finite-dimensional vector space are Banach spaces.

(iv) Finally, in other words, by the definition of Banach space (which be defined as "complete normed vector space"), we can conclude that "every finite-dimensional (normed) vector space is (real / complex) Banach space and which is complete depend on over real numbers number field."

(Proof 2).

There are some important properties which I will show them in the following pages.

(i) Any two norms on a finite dimensional space are equivalent.

i.e. There exist some constants $c > 0$, and $c' > 0$ such that

$$c\|v\|_1 \leq \|v\|_2 \leq c'\|v\|_1$$

for all vectors v , where $\|\cdot\|_1$ and $\|\cdot\|_2$ denote two different norm in this norm vector space $(V, \|\cdot\|_i)$.

$$i \in I = \{1, 2, \dots\}$$

(ii) Any finite-dimensional normed vector space is a Banach space (i.e. is complete)

Since (i) and (ii) are equivalent statement, in other words we can start from (i) to derive (ii) and vice versa. However, (i) \Rightarrow (ii) is need to use more technique in topology and real analysis, so I just write down the reference what I have found

→ "A course in functional analysis" by John. B. Conway
P. 69 ~ 70.

And I will try to prove (ii) first, and then briefly to show how to derive (i) by topological method.

I will prove (i) in a special case, and extend to
 (specific norm)
 general case by using the theorem of all norms
 are equivalent in finite-dimensional normed
 vector space."

Proof:
 Suppose that V is a finite dimensional vector space
 over \mathbb{H} , with basis $\{v_1, \dots, v_r\}$ and norm $\|\cdot\|_0 : V \rightarrow \mathbb{R}$,
 defined for every $x = \sum_{i=1}^r c_i v_i$, where \mathbb{H} is a
 field and can be real numbers field, \mathbb{R} , or
 complex numbers field, \mathbb{C} . And, assume the

$$\|x\|_0 = \left(\sum_{i=1}^r |c_i|^2 \right)^{1/2}$$

We also suppose that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy
 sequence in V . For each $n \in \mathbb{N}$, let $x_n = c_{n1} v_1 + \dots + c_{nr} v_r$,
 where $c_{n1}, \dots, c_{nr} \in \mathbb{H}$ are unique.

Since we can define a metric function

$\rho : V \times V \rightarrow \mathbb{R}$
 and equal to norm:

$$\rho(x_m, x_n) = \|x_m - x_n\|_0$$

$$= \left(\sum_{i=1}^r |c_{mi} - c_{ni}|^2 \right)^{1/2}$$

So that for every fixed $i=1, \dots, r$ we have

$$|c_{ni} - g_i| \leq p_0(x_n, x)$$

and so $(c_{ni})_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{H} .

Since \mathbb{H} is complete, it follows that for every fixed $i=1, \dots, r$ there exists $c_i \in \mathbb{H}$

such that $c_{ni} \rightarrow c_i$ as $n \rightarrow \infty$,

therefore, given any $\epsilon > 0$, there exists N_i such

that $|c_{ni} - c_i| < \frac{\epsilon}{\sqrt{r}}$ whenever $n > N_i$.

So that let $x = c_1 v_1 + c_2 v_2 + \dots + c_r v_r$.

Then we can obtain that

$$\begin{aligned} p_0(x_n, x) &= \|x_n - x\|_0 \\ &= \left(\sum_{i=1}^r |c_{ni} - c_i|^2 \right)^{1/2} < \epsilon \end{aligned}$$

whenever $n > \max\{N_1, \dots, N_r\}$.

It follows that $x_n \rightarrow x$ as $n \rightarrow \infty$. Hence V is

complete.

Now, I will extend this result to any norm

in finite-dimensional normed vector space.

Since we have a theorem: Every finite dimensional normed vector space over \mathbb{H} , \mathbb{H} is complete, with respect

to the basis $\{v_1, v_2, \dots, v_r\}$. Then any norm $\|\cdot\|: V \rightarrow \mathbb{R}$ is equivalent to the norm $\|\cdot\|_0: V \rightarrow \mathbb{R}$.

Thus, we complete the proof that in any

finite-dimensional normed vector space,

the space is complete, i.e. Banach-space.

real or complex (\mathbb{R} or \mathbb{C})
depends on which field the v_i is
over on it

Ex5. By the definition of the inner product in our Lecture note (1.3b) and property of vector space "(1.1h)", we can show this is valid.

Suppose $\psi = \emptyset$ and $a = 0$ number zero.
null vector, notation with capital zero!

(i) Then by (1.3b) we have $(\psi, a\phi) = a(\psi, \phi)$

$$\Rightarrow \text{right hand side} = a(\psi, \phi)$$

$$= 0(\psi, \phi)$$

$$= 0 - 0$$

$$\text{left hand side: } (\psi, a\phi) \underset{\uparrow}{=} (\psi, 0) \quad -②$$

$$\therefore a\phi \underset{\uparrow}{=} 0$$

$$\therefore \text{l.h.s} = \text{r.h.s.} \text{ (i.e., } 0 = 0) \quad \text{by (1.1h)}$$

$$\text{Thus. } (\psi, 0) = 0$$

$$(ii) \therefore (\psi, a\psi) = a(\psi, \psi) = \underset{a=0}{(\psi, \psi)} = (0, \psi) \quad -③$$

$$\text{and } (\psi, a\psi) = (\psi, 0) \quad -④$$

Therefore, by $③ = ④$, we obtain that.

$$(\psi, 0) = (0, \psi)$$

and according (i), we conclude that.

$$(\psi, 0) = (0, \psi) = 0$$

Ex. 6 (i) Show that in $L^2(-1, 1)$ space, norm can be defined as $\|f(x)\| = \sqrt{(f(x), f(x))}$.

(ii) And, also can be defined as

$$\|f(x)\| = \max \{|f(x)|, a \leq x \leq b\}.$$

(Pf i) ① $\|f(x)\| = \sqrt{(f(x), f(x))} \geq 0 \quad (\because \sqrt{a} \geq 0, a \in \mathbb{R})$

$\|f(x)\| = 0 \text{ iff } f(x) \text{ is null.}$

$$② \|af(x)\| = \sqrt{(af(x), af(x))}$$

$$= \sqrt{a^2(f(x), f(x))}$$

$$= |a| \sqrt{(f(x), f(x))}$$

$$= |a| \|f(x)\|$$

$$③ \|f_1(x) + f_2(x)\|$$

$$= \sqrt{(f_1(x) + f_2(x), f_1(x) + f_2(x))}$$

$$= \sqrt{(f_1(x), f_1(x)) + (f_1(x), f_2(x)) + (f_2(x), f_1(x)) + (f_2(x), f_2(x))}$$

$$\Rightarrow \|f_1(x) + f_2(x)\|^2 = (f_1(x), f_1(x)) + \underbrace{(f_1(x), f_2(x))}_{\int_{-1}^1 f_1 f_2 dx} + \underbrace{(f_2(x), f_1(x))}_{\int_{-1}^1 f_1 f_2 dx} + (f_2(x), f_2(x))$$

$$\leq \|f_1(x)\|^2 + \|f_2(x)\|^2 + 2 \int_{-1}^1 |f_1(x)f_2(x)| dx$$

$$\leq (\|f_1(x)\| + \|f_2(x)\|)^2$$

Because by the Schwarz inequality

$$\int_{-1}^1 |f_1(x)f_2(x)| dx \leq \left(\int_{-1}^1 |f_1(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{-1}^1 |f_2(x)|^2 dx \right)^{\frac{1}{2}} = \|f_1(x)\| \|f_2(x)\|$$

$$\text{Finally, } \|f_1\|^2 + \|f_2\|^2 + 2 \int_{-1}^1 |f_1| |f_2| dx \leq (\|f_1\| + \|f_2\|)^2$$

$$\text{that is, } \|f_1 + f_2\|^2 \leq (\|f_1\| + \|f_2\|)^2$$

$$\Rightarrow \|f_1(x) + f_2(x)\| \leq \|f_1(x)\| + \|f_2(x)\|.$$

Therefore, $\|f(x)\| = \sqrt{(f(x), f(x))}$ is a norm
is $L^2(-1, 1)$ space.

(Proof of (ii))

① since $|f(x)|$ always positive or larger/equal

to zero, $|f(x)| \geq 0,$

$(|f(x)| = 0 \text{ iff } f(x) = 0)$

thus, $\max \{|f(x)|, a \leq x \leq b\} \geq 0.$

and then $\|f(x)\| \geq 0. \rightarrow \text{condition ① is valid!}$

$$\textcircled{2} \|a'f(x)\| = \max \left\{ |a'f(x)|, \underset{\substack{\uparrow \\ \text{note that}}}{a \leq x \leq b} \right\}.$$

$a' \neq a$

$$= \max \{ |a'| |f(x)|, a \leq x \leq b \}$$

$$= |a'| \cdot \max \{ |f(x)|, a \leq x \leq b \}$$

$$= |a'| \cdot \|f(x)\|. \rightarrow \text{Condition ② is valid!}$$

$$\textcircled{3} \|f_1(x) + f_2(x)\| = \max \{ |f_1(x) + f_2(x)|, a \leq x \leq b \}.$$

$$\Rightarrow \max \left\{ |f_1(x) + f_2(x)|, x \in [a, b] \right\} \leq \max_{x \in [a, b]} \left\{ |f_1(x)| + |f_2(x)| \right\}$$

$$\leq \underbrace{\max_{x \in [a, b]} \left\{ |f_1(x)| \right\}}_{\text{i.e. } \|f_1(x)\|} + \underbrace{\max_{x \in [a, b]} \left\{ |f_2(x)| \right\}}_{= \|f_2(x)\|}$$

It follows that, the condition ③ is also valid for

the maximum norm : $\|f(x)\| = \max \left\{ |f(x)|, a \leq x \leq b \right\}$

or write it in another notation to more briefly

$$\Rightarrow \|f(x)\| = \max_{x \in [a, b]} \left\{ |f(x)| \right\}$$

which is a norm in $L^2(-1, 1)$

space.