Consider the following set of continuous functions, \( \{f_n(x) = x^n \mid x \in [-1, 1]\}_{n \in \mathbb{N}} \), which span \( L^2(-1, 1) \). Explicitly determine the first three orthonormal functions by using the Gram-Schmidt orthonormalization process. The results remind you of which functions?

**Solution.** To begin we find the first three orthogonal basis vectors by following the procedure outlined on page 5:

\[
f_0'(x):
\]
\[
f_0'(x) = f_0(x) = 1
\]

\[
f_1'(x):
\]
\[
f_1'(x) = f_1(x) - f_0'(x)\frac{(f_0'(x), f_1(x))}{(f_0'(x), f_0'(x))} = x - \frac{(1, x)}{(1, 1)}
\]
\[= x - \frac{x^2|_{-1}^{1}}{x|_{-1}^{1}} = x - \frac{1 - \frac{1}{2}}{1 - (-1)} = x
\]

\[
f_2'(x):
\]
\[
f_2'(x) = f_2(x) - f_1'(x)\frac{(f_1'(x), f_2(x))}{(f_1'(x), f_1'(x))} - f_0'(x)\frac{(f_0'(x), f_2(x))}{(f_0'(x), f_0'(x))}
\]
\[= x^2 - x \cdot \frac{(x, x^2)}{(x, x)} - (1, x^2) - \frac{x^3|_{-1}^{1}}{x|_{-1}^{1}} = x^2 - x \cdot \frac{1 - \frac{1}{4}}{1 - (-1)} - \frac{1}{3} - (\frac{-1}{3}) = x^2 - \frac{1}{3}
\]

From here we normalize the basis by dividing by the magnitude:

\[
\tilde{f}_0(x) = \frac{f_0'(x)}{\sqrt{(f_0'(x), f_0'(x))}} = \frac{1}{\sqrt{1}} = 1
\]

\[
\tilde{f}_1(x) = \frac{f_1'(x)}{\sqrt{(f_1'(x), f_1'(x))}} = \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} = \frac{x}{\sqrt{x^3|_{-1}^{1}}} = x\frac{3}{2}
\]

\[
\tilde{f}_2(x) = \frac{f_2'(x)}{\sqrt{(f_2'(x), f_2'(x))}} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx}} = \frac{x^2 - \frac{1}{3}}{\sqrt{3\int_{-1}^{1} (x^4 - \frac{2}{3}x^2 + \frac{1}{9}) dx}}
\]
\[= \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{3}{5} - \frac{2}{9}x^3 + \frac{1}{9}x}|_{-1}^{1}} = \sqrt{\frac{3}{5} - \frac{2}{9}x^3 + \frac{1}{9}x - (-\frac{3}{5} + \frac{2}{9} - \frac{1}{9})} = (3x^2 - 1)\frac{\sqrt{5}}{8}
\]
These are proportional to the Legendre polynomials which occur, for example, when solving the Laplace equation for the Newtonian potential.

**Ex. 2** Prove the Minkowski inequality holds in Hilbert space, i.e.

\[ \|\psi + \phi\| \leq \|\psi\| + \|\phi\|. \]

[Hint: Compute the square of each side]

**Proof.** As the hint indicates, we begin by computing

\[ (\|\psi\| + \|\phi\|)^2 = \|\psi\|^2 + 2\|\psi\|\|\phi\| + \|\phi\|^2 \]

and

\[ \|\psi + \phi\|^2 = (\psi + \phi, \psi + \phi) = (\psi + \phi, \psi) + (\psi + \phi, \phi) \]
\[ = (\psi, \psi) + (\phi, \psi) + (\psi, \phi) + (\phi, \phi) = \|\psi\|^2 + (\psi, \phi)^* + (\psi, \phi) + \|\phi\|^2 \]
\[ = \|\psi\|^2 + 2\text{Re}(\psi, \phi) + \|\phi\|^2 \]

Thus, by applying the Schwarz inequality (1.2.1) we have:

\[ \|\psi + \phi\|^2 = \|\psi\|^2 + 2\text{Re}(\psi, \phi) + \|\phi\|^2 \leq \|\psi\|^2 + 2\|\psi\|\|\phi\| + \|\phi\|^2 = (\|\psi\| + \|\phi\|)^2 \]

By taking square roots on both sides, and using the fact that \( f(x) = x^2 \) is monotonically increasing on \((0, \infty)\), we have the desired result.

**Ex. 3** Prove the parallelogram law holds in Hilbert space, i.e.

\[ \|\psi + \phi\|^2 + \|\psi - \phi\|^2 = 2(\|\psi\|^2 + \|\phi\|^2) \]

**Proof.** Recall from the proof of Ex. 2 that we computed

\[ \|\psi + \phi\|^2 = \|\psi\|^2 + 2\text{Re}(\psi, \phi) + \|\phi\|^2. \]

Similarly, we have that

\[ \|\psi - \phi\|^2 = \|\psi\|^2 - 2\text{Re}(\psi, \phi) + \|\phi\|^2. \]

By adding these two equations we have

\[ \|\psi + \phi\|^2 + \|\psi - \phi\|^2 = (\|\psi\|^2 + 2\text{Re}(\psi, \phi) + \|\phi\|^2) + (\|\psi\|^2 - 2\text{Re}(\psi, \phi) + \|\phi\|^2) \]
\[ = 2\|\psi\|^2 + 2\|\phi\|^2 = 2(\|\psi\|^2 + \|\phi\|^2), \]

as desired.
**Ex. 4** Prove that every finite-dimensional vector space is complete.
[Hint: Recall that both the real and complex numbers are complete]

*Proof.* Let $X$ be a vector space of dimension $n < \infty$. Denote the basis of $X$ by $\{\phi_1, \cdots, \phi_n\}$ and the basis of $\mathbb{R}^n$ by $\{e_1, \cdots, e_n\}$. Consider the canonical map $\text{Can}_X : X \to \mathbb{R}^n$, defined by

$$\text{Can}_X \left( \sum_{i=1}^{n} a_i \phi_i \right) = \sum_{i=1}^{n} a_i e_i.$$  

We will make free use of two standard results about $\text{Can}_X$--it is an isomorphism of vector spaces, as well as a homeomorphism between the underlying norm-induced topologies. Given a Cauchy sequence $\{b_i\}$ in $X$ we have that

$$\lim_{i \to \infty} b_i = \lim_{i \to \infty} \text{Can}_X^{-1}(a_i) = \text{Can}_X^{-1}(\lim_{i \to \infty} a_i) = \text{Can}_X^{-1}(a) = b,$$

where the limit $a$ of $\{a_i\}$ exists since $\mathbb{R}^n$ is complete. Therefore every Cauchy sequence in $X$ converges, i.e. $X$ is complete. \hfill \Box

**Ex. 5** By using the definition of the inner product in $V$, prove that $(\psi, 0) = (0, \psi) = 0$.

*Proof.* We have from 1.3a that

$$(\psi, 0) = (\psi, 0 + 0) = (\psi, 0) + (\psi, 0) = 2(\psi, 0).$$

Subtracting $(\psi, 0)$ from both sides yields

$$0 = (\psi, 0).$$

The other equality follows from the above and 1.3c, since we have

$$(0, \psi) = (\psi, 0)^* = 0^* = 0.$$

\hfill \Box

**Ex. 6** Show that both $\|f\|_1 = \sqrt{(f(x), f(x))}$ and $\|f\|_2 = \max_{a \leq x \leq b} |f(x)|$ define a norm on $L^2(a, b)$.

*Proof.* First we will verify 1.19a-c for $\| \cdot \|_1$:

1.19a: Since $|f(x)|^2 > 0$ we have that

$$\langle f(x), f(x) \rangle = \int_a^b f^*(x)f(x) \, dx = \int_a^b |f(x)|^2 \, dx > 0;$$

and therefore, since $\sqrt{\cdot} : (0, \infty) \to (0, \infty), \|f(x)\|_1 = \sqrt{\langle f(x), f(x) \rangle} > 0.$
This follows from a simple calculation:

\[ \|af(x)\|_1 = \sqrt{(af(x), af(x))} = \left( \int_a^b |af(x)|^2 \, dx \right)^{1/2} = \left( a^2 \int_a^b |f(x)|^2 \, dx \right)^{1/2} \]

\[ = (a^2)^{1/2} \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2} = |a| \cdot \|f(x)\|_1, \]

where the third equality follows from the fact that \(|\cdot|\) is a norm on \(\mathbb{R}\).

1.19c: This follows from the argument given in Ex. 2, but we will reproduce it here in this context. Given \(f, g \in L^2(a, b)\) we have that

\[ (\|f(x)\|_1 + \|g(x)\|_1)^2 = \|f(x)\|^2_1 + 2\|f(x)\|_1 \|g(x)\|_1 + \|g(x)\|^2_1 \]

and

\[ \|f(x) + g(x)\|^2_1 = (f(x) + g(x), f(x) + g(x)) = \int_a^b (f(x) + g(x))^* (f(x) + g(x)) \, dx \]

\[ = \int_a^b (f(x) + g(x))^* (f(x) + g(x)) \, dx = \int_a^b (f(x)^* f(x) + f(x)^* g(x) + g(x)^* f(x) + g(x) g(x)^*) \, dx \]

\[ = \int_a^b f(x)^* f(x) \, dx + \int_a^b g(x)^* g(x) \, dx + \int_a^b f(x)^* g(x) + g(x)^* f(x) \, dx \]

\[ = (f(x), f(x))^2 + 2 \int_a^b \text{Re}(f(x)^* g(x)) \, dx + (g(x), g(x))^2 \]

\[ = \|f(x)\|^2_1 + 2 \text{Re}(f(x), g(x)) + \|g(x)\|^2_1. \]

Thus, by applying the Schwarz inequality (1.2.1) we have:

\[ \|f(x) + g(x)\|^2_1 = \|f(x)\|^2_1 + 2 \text{Re}(f(x), g(x)) + \|g(x)\|^2_1 \]

\[ \leq \|f(x)\|^2_1 + 2\|f(x)\|_1 \|g(x)\|_1 + \|g(x)\|^2_1 = (\|f(x)\|_1 + \|g(x)\|_1)^2 \]

By taking square roots on both sides, and using the fact that \(f(x) = x^2\) is monotonically increasing on \((0, \infty)\), we have the established that \(\|\cdot\|_1\) satisfies the triangle inequality; thus it is a norm on \(L^2(a, b)\).

Next we will verify 1.19a-c for \(\|\cdot\|_2\):

1.19a: Since \(|f(x)| \geq 0\) it is vacuously true that \(\max_{a \leq x \leq b} |f(x)| \geq 0\). Furthermore, since \(|\cdot|\) is a norm on \(\mathbb{R}\) we have that

\[ \|f\|_2 = \max_{a \leq x \leq b} |f(x)| = 0 \iff |f(x)| = 0, \quad \forall \ x \in [a, b] \iff f(x) \equiv 0 \text{ on } [a, b]. \]
1.19b: \[ \|\alpha f(x)\|_2 = \max_{a \leq x \leq b} |\alpha f(x)| = \max_{a \leq x \leq b} |\alpha||f(x)| = |\alpha| \max_{a \leq x \leq b} |f(x)| = |\alpha||f(x)||_2. \]

1.19c: To prove the triangle inequality let \( x_0 \in [a, b] \) be such that \( |f(x_0) + g(x_0)| = \max_{a \leq x \leq b} |f(x) + g(x)| \). Then by definition of the maximum we have that for every \( x \in [a, b] \)

\[ |f(x) + g(x)| \leq |f(x_0) + g(x_0)| \leq |f(x_0)| + |g(x_0)| \leq \max_{a \leq x \leq b} |f(x)| + \max_{a \leq x \leq b} |g(x)|, \]

where the second inequality is an application of the triangle inequality for \(|\cdot|\), and the third inequality is by the definition of \( \max(\cdot) \). Since this is true for every \( x \in [a, b] \) we have that

\[ \max_{a \leq x \leq b} |f(x) + g(x)| \leq \max_{a \leq x \leq b} |f(x)| + \max_{a \leq x \leq b} |g(x)|. \]