$\boxed{\text{Ex. 1}}$ Consider the following set of continuous functions, $\{f_n(x) = x^n \mid x \in [-1,1]\}_{n\in\mathbb{N}}$, which span $\mathcal{L}^2(-1,1)$. Explicitly determine the first three orthonormal functions by using the Gram-Schmidt orthonormalization process. The results remind you of which functions?

Solution. To begin we find the first three orthogonal basis vectors by following the procedure outlined on page 5:

$f_0'(x)$:

$$f_0'(x) = f_0(x) = 1$$

$f_1'(x)$:

$$f_1'(x) = f_1(x) - f_0'(x)\frac{(f_0'(x), f_1(x))}{(f_0'(x), f_0'(x))} = x - \frac{(1,x)}{(1,1)}$$

$$= x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} 1\,dx} = x - \frac{\frac{x^2}{2}\Big|_{-1}^{1}}{x|_{-1}^{1}} = x - \frac{\frac{1}{2}-\frac{1}{2}}{1-(-1)} = x$$

$f_2'(x)$:

$$f_2'(x) = f_2(x) - f_1'(x)\frac{(f_1'(x), f_2(x))}{(f_1'(x), f_1'(x))} - f_0'(x)\frac{(f_0'(x), f_2(x))}{(f_0'(x), f_0'(x))}$$

$$= x^2 - x\cdot\frac{(x,x^2)}{(x,x)} - \frac{(1,x^2)}{(1,1)} = x^2 - x\cdot\frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx}$$

$$= x^2 - x\cdot\frac{\frac{x^4}{4}\Big|_{-1}^{1}}{\frac{x^3}{3}\Big|_{-1}^{1}} - \frac{\frac{x^3}{3}\Big|_{-1}^{1}}{x|_{-1}^{1}} = x^2 - x\cdot\frac{\frac{1}{4}-\frac{1}{4}}{\frac{1}{3}-\left(-\frac{1}{3}\right)} - \frac{\frac{1}{3}-\left(-\frac{1}{3}\right)}{1-(-1)} = x^2 - \frac{1}{3}$$

From here we normalize the basis by dividing by the magnitude:

$$\widetilde{f_0}(x) = \frac{f_0'(x)}{\sqrt{(f_0'(x), f_0'(x))}} = \frac{1}{\sqrt{\int_{-1}^{1} 1\,dx}} = \frac{1}{\sqrt{x|_{-1}^{1}}} = \frac{1}{\sqrt{2}}$$

$$\widetilde{f_1}(x) = \frac{f_1'(x)}{\sqrt{(f_1'(x), f_1'(x))}} = \frac{x}{\sqrt{\int_{-1}^{1} x^2\,dx}} = \frac{x}{\sqrt{x^3/3|_{-1}^{1}}} = x\sqrt{\frac{3}{2}}$$

$$\widetilde{f_2}(x) = \frac{f_2'(x)}{\sqrt{(f_2'(x), f_2'(x))}} = \frac{x^2-\frac{1}{3}}{\sqrt{\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2\,dx}} = \frac{x^2-\frac{1}{3}}{\sqrt{\int_{-1}^{1}(x^4-\frac{2}{3}x^2+\frac{1}{9})\,dx}}$$

$$= \frac{x^2-\frac{1}{3}}{\sqrt{(\frac{x^5}{5}-\frac{2}{9}x^3+\frac{1}{9}x)|_{-1}^{1}}} = \frac{\frac{3x^2-1}{3}}{\sqrt{(\frac{1}{5}-\frac{2}{9}+\frac{1}{9})-(-\frac{1}{5}+\frac{2}{9}-\frac{1}{9})}} = (3x^2-1)\sqrt{\frac{5}{8}}$$

These are proportional to the Legendre polynomials which occur, for example, when solving the Laplace equation for the Newtonian potential.

Ex. 2 Prove the Minkowski inequality holds in Hilbert space, i.e.

$$\|\psi + \phi\| \leq \|\psi\| + \|\phi\|.$$

[Hint: Compute the square of each side]

Proof. As the hint indicates, we begin by computing

$$(\|\psi\| + \|\phi\|)^2 = \|\psi\|^2 + 2\|\psi\|\|\phi\| + \|\phi\|^2$$

and

$$\begin{aligned} \|\psi + \phi\|^2 = (\psi + \phi, \psi + \phi) = (\psi + \phi, \psi) + (\psi + \phi, \phi) \\ = (\psi, \psi) + (\phi, \psi) + (\psi, \phi) + (\phi, \phi) = \|\psi\|^2 + (\psi, \phi)^* + (\psi, \phi) + \|\phi\|^2 \\ = \|\psi\|^2 + 2\,\mathrm{Re}(\psi, \phi) + \|\phi\|^2 \end{aligned}$$

Thus, by applying the Schwarz inequality (1.2.1) we have:

$$\|\psi + \phi\|^2 = \|\psi\|^2 + 2\,\mathrm{Re}(\psi, \phi) + \|\phi\|^2 \leq \|\psi\|^2 + 2\|\psi\|\|\phi\| + \|\phi\|^2 = (\|\psi\| + \|\phi\|)^2$$

By taking square roots on both sides, and using the fact that $f(x) = x^2$ is monotonically increasing on $(0, \infty)$, we have the desired result. $\square$

Ex. 3 Prove the parallelogram law holds in Hilbert space, i.e.

$$\|\psi + \phi\|^2 + \|\psi - \phi\|^2 = 2(\|\psi\|^2 + \|\phi\|^2)$$

Proof. Recall from the proof of Ex. 2 that we computed

$$\|\psi + \phi\|^2 = \|\psi\|^2 + 2\,\mathrm{Re}(\psi, \phi) + \|\phi\|^2.$$

Similarly, we have that

$$\|\psi - \phi\|^2 = \|\psi\|^2 - 2\,\mathrm{Re}(\psi, \phi) + \|\phi\|^2.$$

By adding these two equations we have

$$\begin{aligned} \|\psi + \phi\|^2 + \|\psi - \phi\|^2 = (\|\psi\|^2 + \cancel{2\,\mathrm{Re}(\psi, \phi)} + \|\phi\|^2) + (\|\psi\|^2 - \cancel{2\,\mathrm{Re}(\psi, \phi)} + \|\phi\|^2) \\ = 2\|\psi\|^2 + 2\|\phi\|^2 = 2(\|\psi\|^2 + \|\phi\|^2), \end{aligned}$$

as desired. $\square$

$\boxed{\text{Ex. 4}}$ Prove that every finite-dimensional vector space is complete.
[Hint: Recall that both the real and complex numbers are complete]

Proof. Let $\mathcal{X}$ be a vector space of dimension $n < \infty$. Denote the basis of $\mathcal{X}$ by $\{\phi_1, \cdots, \phi_n\}$ and the basis of $\mathbb{R}^n$ by $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$. Consider the canonical map $\mathbb{C}\text{an}_{\mathcal{X}} : \mathcal{X} \to \mathbb{R}^n$, defined by

$$\mathbb{C}\text{an}_{\mathcal{X}} \left(\sum_{i=1}^{n} a_i \phi_i \right) = \sum_{i=1}^{n} a_i \mathbf{e}_i.$$

We will make free use of two standard results about $\mathbb{C}\text{an}_{\mathcal{X}}$- it is an isomorphism of vector spaces, as well as a homeomorphism between the underlying norm-induced topologies. Given a Cauchy sequence $\{b_i\}$ in $\mathcal{X}$ we have that

$$\lim_{i\to\infty} b_i = \lim_{i\to\infty} \mathbb{C}\text{an}^{-1}_{\mathcal{X}}(a_i) = \mathbb{C}\text{an}^{-1}_{\mathcal{X}}(\lim_{i\to\infty} a_i) = \mathbb{C}\text{an}^{-1}_{\mathcal{X}}(a) = b,$$

where the limit a of $\{a_i\}$ exists since $\mathbb{R}^n$ is complete. Therefore every Cauchy sequence in $\mathcal{X}$ converges, i.e. $\mathcal{X}$ is complete. □

$\boxed{\text{Ex. 5}}$ By using the definition of the inner product in $\mathcal{V}$, prove that $(\psi, 0) = (0, \psi) = 0$.

Proof. We have from 1.3a that

$$(\psi, 0) = (\psi, 0 + 0) = (\psi, 0) + (\psi, 0) = 2(\psi, 0).$$

Subtracting $(\psi, 0)$ from both sides yields

$$0 = (\psi, 0).$$

The other equality follows from the above and 1.3c, since we have

$$(0, \psi) = (\psi, 0)^* = 0^* = 0.$$

□

$\boxed{\text{Ex. 6}}$ Show that both $\|f\|_1 = \sqrt{(f(x), f(x))}$ and $\|f\|_2 = \max_{a\leq x\leq b} |f(x)|$ define a norm on $\mathcal{L}^2(a, b)$.

Proof. First we will verify 1.19a-c for $\|\cdot\|_1$:

<u>1.19a:</u> Since $|f(x)|^2 > 0$ we have that

$$(f(x), f(x)) = \int_a^b f^*(x) f(x)\, dx = \int_a^b |f(x)|^2\, dx > 0;$$

and therefore, since $\sqrt{\cdot} : (0, \infty) \to (0, \infty)$, $\|f(x)\|_1 = \sqrt{(f(x), f(x))} > 0$.

1.19b: This follows from a simple calculation:

$$\|af(x)\|_1 = \sqrt{(af(x), af(x))} = \left(\int_a^b |af(x)|^2\,dx\right)^{1/2} = \left(a^2\int_a^b |f(x)|^2\,dx\right)^{1/2}$$

$$= (a^2)^{1/2}\left(\int_a^b |f(x)|^2\,dx\right)^{1/2} = |a|\cdot\|f(x)\|_1,$$

where the third equality follows from the fact that $|\cdot|$ is a norm on $\mathbb{R}$.

1.19c: This follows from the argument given in Ex. 2, but we will reproduce it here in this context. Given $f, g \in \mathcal{L}^2(a,b)$ we have that

$$(\|f(x)\|_1 + \|g(x)\|_1)^2 = \|f(x)\|_1^2 + 2\|f(x)\|_1\|g(x)\|_1 + \|g(x)\|_1^2$$

and

$$\|f(x)+g(x)\|_1^2 = (f(x)+g(x), f(x)+g(x)) = \int_a^b (f(x)+g(x))^*(f(x)+g(x))\,dx$$

$$= \int_a^b (f(x)^*+g(x)^*)(f(x)+g(x))\,dx = \int_a^b (f(x)^*f(x)+f(x)^*g(x)+g(x)^*f(x)+g(x)g(x)^*)\,dx$$

$$= \int_a^b f(x)^*f(x)\,dx + \int_a^b [f(x)^*g(x) + (f(x)^*g(x))^*]\,dx + \int_a^b g(x)g(x)^*\,dx$$

$$= (f(x), f(x))^2 + 2\int_a^b \mathrm{Re}(f(x)^*g(x))\,dx + (g(x), g(x))^2$$

$$= \|f(x)\|_1^2 + 2\,\mathrm{Re}(f(x), g(x)) + \|g(x)\|_1^2.$$

Thus, by applying the Schwarz inequality (1.2.1) we have:

$$\|f(x)+g(x)\|_1^2 = \|f(x)\|_1^2 + 2\,\mathrm{Re}(f(x), g(x)) + \|g(x)\|_1^2$$

$$\le \|f(x)\|_1^2 + 2\|f(x)\|_1\|g(x)\|_1 + \|g(x)\|_1^2 = (\|f(x)\|_1 + \|g(x)\|_1)^2$$

By taking square roots on both sides, and using the fact that $f(x) = x^2$ is monotonically increasing on $(0,\infty)$, we have the established that $\|\cdot\|_1$ satisfies the triangle inequality; thus it is a norm on $\mathcal{L}^2(a,b)$.

Next we will verify 1.19a-c for $\|\cdot\|_2$:

1.19a: Since $|f(x)| \ge 0$ it is vacuously true that $\max_{a\le x\le b}|f(x)| \ge 0$. Furthermore, since $|\cdot|$ is a norm on $\mathbb{R}$ we have that

$$\|f\|_2 = \max_{a\le x\le b}|f(x)| = 0 \quad\Leftrightarrow\quad |f(x)| = 0,\ \forall x\in[a,b] \quad\Leftrightarrow\quad f(x)\equiv 0 \text{ on } [a,b].$$

1.19b:

$$\|\alpha f(x)\|_2 = \max_{a\leq x\leq b} |\alpha f(x)| = \max_{a\leq x\leq b} |\alpha||f(x)| = |\alpha| \cdot \max_{a\leq x\leq b} |f(x)| = |\alpha|\|f(x)\|_2.$$

1.19c: To prove the triangle inequality let $x_0 \in [a,b]$ be such that $|f(x_0) + g(x_0)| = \max_{a\leq x\leq b} |f(x) + g(x)|$. Then by definition of the maximum we have that for every $x \in [a,b]$

$$|f(x) + g(x)| \leq |f(x_0) + g(x_0)| \leq |f(x_0)| + |g(x_0)| \leq \max_{a\leq x\leq b} |f(x)| + \max_{a\leq x\leq b} |g(x)|,$$

where the second inequality is an application of the triangle inequality for $|\cdot|$, and the third inequality is by the definition of $\max(\cdot)$. Since this is true for every $x \in [a,b]$ we have that

$$\max_{a\leq x\leq b} |f(x) + g(x)| \leq \max_{a\leq x\leq b} |f(x)| + \max_{a\leq x\leq b} |g(x)|.$$

$\square$