$\boxed{\text{Ex. 1}}$ Find the quantum uncertainty $\Delta\xi\Delta P_\xi$ in the n particle state (or the n-th excited state) of the one-dimensional oscillator.

Solution.

This follows from a few short calculations. By definition we have that

$$(\Delta\,\xi)^2 = \left\langle \xi^2 \right\rangle - \langle \xi \rangle^2 \qquad \text{and} \qquad (\Delta P_\xi)^2 = \left\langle P_\xi^2 \right\rangle - \langle P_\xi \rangle^2 .$$

Using (2.57), (2.58), (2.65), and (2.66) can calculate the relevant quantities as follows:

$\underline{\langle \xi \rangle :}$

$$\begin{aligned}\langle \xi \rangle &= \frac{1}{\sqrt{2}}(\langle n|\,a\,|n\rangle + \langle n|\,a^\dagger\,|n\rangle) \\ &= \frac{1}{\sqrt{2}}(\cancel{\sqrt{n}\,\langle n\,|n-1\rangle} + \cancel{\sqrt{n+1}\,\langle n\,|n+1\rangle}) = 0;\end{aligned}$$

$\underline{\langle \xi^2 \rangle :}$

$$\begin{aligned}\left\langle \xi^2 \right\rangle &= \frac{1}{2}(\langle n|\,a^2\,|n\rangle + \langle n|\,aa^\dagger\,|n\rangle + \langle n|\,a^\dagger a\,|n\rangle + \langle n|\,(a^\dagger)^2\,|n\rangle) \\ &= \frac{1}{2}(\sqrt{n}\,\langle n|\,a\,|n-1\rangle + \sqrt{n+1}\,\langle n|\,a\,|n+1\rangle + \sqrt{n}\,\langle n|\,a^\dagger\,|n-1\rangle + \sqrt{n+1}\,\langle n|\,a^\dagger\,|n+1\rangle) \\ &= \frac{1}{2}(\cancel{\sqrt{n(n-1)}\,\langle n\,|n-2\rangle} + (n+1)\,\langle n\,|n\rangle + n\,\langle n\,|n\rangle + \cancel{\sqrt{(n+1)(n+2)}\,\langle n\,|n+2\rangle}) \\ &= \frac{1}{2}(2n+1) = n + \frac{1}{2};\end{aligned}$$

$\underline{\langle P_\xi \rangle :}$

$$\begin{aligned}\langle P_\xi \rangle &= \frac{i}{\sqrt{2}}(\langle n|\,a^\dagger\,|n\rangle - \langle n|\,a\,|n\rangle) \\ &= \frac{i}{\sqrt{2}}(\cancel{\sqrt{n+1}\,\langle n\,|n+1\rangle} - \cancel{\sqrt{n}\,\langle n\,|n-1\rangle}) = 0;\end{aligned}$$

$\underline{\langle P_\xi^2 \rangle :}$

$$\begin{aligned}\left\langle P_\xi^2 \right\rangle &= -\frac{1}{2}(\langle n|\,(a^\dagger)^2\,|n\rangle - \langle n|\,a^\dagger a\,|n\rangle - \langle n|\,aa^\dagger\,|n\rangle + \langle n|\,a^2\,|n\rangle) \\ &= -\frac{1}{2}(\sqrt{n+1}\,\langle n|\,a^\dagger\,|n+1\rangle - \sqrt{n}\,\langle n|\,a^\dagger\,|n-1\rangle - \sqrt{n+1}\,\langle n|\,a\,|n+1\rangle + \sqrt{n}\,\langle n|\,a\,|n-1\rangle) \\ &= -\frac{1}{2}(\cancel{\sqrt{(n+1)(n+2)}\,\langle n\,|n+2\rangle} - n\,\langle n\,|n\rangle - (n+1)\,\langle n\,|n\rangle + \cancel{\sqrt{n(n-1)}\,\langle n\,|n-2\rangle}) \\ &= -\frac{1}{2}(-2n-1) = n + \frac{1}{2}.\end{aligned}$$

Plugging all of this in where necessary we have

$$\Delta\xi\Delta P_\xi = \sqrt{(\langle\xi^2\rangle - \langle\xi\rangle^2)(\langle P_\xi^2\rangle - \langle P_\xi\rangle^2)} = \sqrt{\left(\left[n+\frac{1}{2}\right] - [0]^2\right)\left(\left[n+\frac{1}{2}\right] - [0]^2\right)}$$
$$= \sqrt{\left(n+\frac{1}{2}\right)^2} = n + \frac{1}{2}.$$

It should be noted that, although the calculations given above will not have the same form in the cases $n = 0$ or $n = 1$ (since the undefined "states" $|-1\rangle$ and $|-2\rangle$ may appear), the same result is produced, as these terms simply disappear earlier in light of (2.61), its conjugate expression, and (2.59). Additionally, note that by rewriting the definitions (see (2.53)) of ξ and P_ξ as

$$\mathbf{X} = \sqrt{\frac{\hbar}{m\omega}}\,\xi \qquad \text{and} \qquad \mathbf{P} = \sqrt{m\omega\hbar}\,P_\xi,$$

we have that

$$\Delta\mathbf{X}\Delta\mathbf{P} = \left(\frac{\hbar}{\cancel{m\omega}}\Delta\xi\right)(\cancel{m\omega}\hbar\Delta P_\xi) = \hbar^2\Delta\xi\Delta P_\xi = \hbar^2\left(n+\frac{1}{2}\right),$$

and, in particular, the ground state uncertainty is

$$\Delta\mathbf{X}\Delta\mathbf{P} = \frac{\hbar^2}{2}.$$

So, the quantum harmonic oscillator actually achieves the *minimum* uncertainty allowed by the Heisenberg uncertainty principle.

$\boxed{\text{Ex. 2}}$ Show that

$$\langle 0|\,e^{ik\mathbf{X}}\,|0\rangle = \exp\left(-\frac{k^2}{2}\,\langle 0|\,\mathbf{X}^2\,|0\rangle\right),$$

where $|0\rangle$ is the ground state of the one-dimensional harmonic oscillator, and **X** is the position operator.

Proof. We begin by defining a new quantity

$$\mu = -ik\sqrt{\frac{\hbar}{2m\omega}}.$$

Then, by (2.53) and (2.56),

$$\mu(a^\dagger + a) = \mu(2\,\mathrm{Re}(a)) = \mu\left(2\sqrt{\frac{m\omega}{2\hbar}}\mathbf{X}\right)$$

$$= 2\left(-ik\sqrt{\frac{\hbar}{2m\omega}}\right)\left(\sqrt{\frac{m\omega}{2\hbar}}\mathbf{X}\right) = \cancel{2}\left(-ik\sqrt{\frac{\cancel{m\hbar\omega}}{\cancel{4m\omega\hbar}}}\mathbf{X}\right) = -ik\mathbf{X},$$

so we have

$$\langle 0| e^{ik\mathbf{X}} |0\rangle = \langle 0| e^{\mu a^\dagger + \mu a} |0\rangle .$$

Then, by using (7) from Lemma 1 of my solutions to Problem Set 3, we have

$$= \langle 0| e^{\mu a^\dagger} e^{\mu a} \exp\left(-\frac{1}{2}[\mu a^\dagger, \mu a]\right) |0\rangle = e^{\langle 0|\mu a^\dagger|0\rangle} e^{\langle 0|\mu a|0\rangle} \exp\left(-\frac{\mu^2}{2}\langle 0| [a^\dagger, a] |0\rangle\right)$$

Next, notice that (2.61) implies that the exponents in the first two terms are zero, so their exponentials are just 1, and these terms disappear. Furthermore, using the anticommutativity of $[\cdot, \cdot]$ on (2.59), we have that $[a^\dagger, a] = -\mathbf{I}$. Putting all of this into the above, we arrive at

$$\langle 0| e^{ik\mathbf{X}} |0\rangle = \exp\left(\frac{1}{2}\left[-ik\sqrt{\frac{\hbar}{2m\omega}}\right]^2 \langle 0| \mathbf{I} |0\rangle\right) = \exp\left(-\frac{k^2\hbar}{4m\omega}\langle 0| \mathbf{I} |0\rangle\right)$$

From here we need to examine the desired matrix element $\langle 0| \mathbf{X}^2 |0\rangle$. Combining (2.53) and (2.57) we have

$$\mathbf{X} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \quad \Rightarrow \quad \mathbf{X}^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2 = \frac{\hbar}{2m\omega}(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2);$$

thus, using (2.61) again, as well as (2.59) and the linearity of the operator above, we have

$$\langle 0| \mathbf{X}^2 |0\rangle = \frac{\hbar}{2m\omega}(\cancel{\langle 0| a^2 |0\rangle} + \langle 0| aa^\dagger |0\rangle + \cancel{\langle 0| a^\dagger a |0\rangle} + \cancel{\langle 0| (a^\dagger)^2 |0\rangle})$$

$$= \frac{\hbar}{2m\omega}\langle 0| aa^\dagger |0\rangle = \frac{\hbar}{2m\omega}(\langle 0| \mathbf{I} |0\rangle - \cancel{\langle 0| a^\dagger a |0\rangle}) = \frac{\hbar}{2m\omega}\langle 0| \mathbf{I} |0\rangle .$$

Substituting this into the expression we derived for the matrix element in question, we have

$$\langle 0| e^{ik\mathbf{X}} |0\rangle = \exp\left(-\frac{k^2}{2}\cdot\frac{\hbar}{2m\omega}\langle 0| \mathbf{I} |0\rangle\right) = \exp\left(-\frac{k^2}{2}\langle 0| \mathbf{X}^2 |0\rangle\right).$$

□

$\boxed{\text{Ex. 3}}$ The Hermite Polynomials $H_n(x)$ may also be defined by the generating function

$$f(h, x) = e^{2hx - h^2} = \sum_n \frac{1}{n!}H_n(x)h^n.$$

Show that the above definition of $H_n(x)$ is consistent with

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}.$$

Proof. Fix x and suppose that we have

$$e^{2hx-h^2} = \sum_n \frac{1}{n!} H_n(x) h^n.$$

Now, consider the Taylor series about 0 for the left-hand side:

$$e^{2hx-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} \left(\frac{d^n}{dh^n} e^{2hx-h^2} \bigg|_{h=0} \right)$$

$$= \sum_{n=0}^{\infty} \frac{h^n}{n!} \left(e^{x^2} e^{-x^2} \frac{d^n}{dh^n} e^{2hx-h^2} \bigg|_{h=0} \right)$$

$$= \sum_{n=0}^{\infty} \frac{h^n}{n!} e^{x^2} \left(\frac{d^n}{dh^n} e^{-x^2+2hx-h^2} \bigg|_{h=0} \right)$$

$$= \sum_{n=0}^{\infty} \frac{h^n}{n!} e^{x^2} \left(\frac{d^n}{dh^n} e^{-(x-h)^2} \bigg|_{h=0} \right)$$

$$= \sum_{n=0}^{\infty} \frac{h^n}{n!} e^{x^2} \left(\frac{n!}{2\pi i} \int_\gamma \frac{e^{-(x-h)^2}}{h^{n+1}} dh \right)$$

(by Cauchy's Integral Formula, where γ is a simple closed curve about the origin)

$$= \sum_{n=0}^{\infty} \frac{h^n}{n!} e^{x^2} \left(\frac{n!}{2\pi i} \int_{\gamma'} \frac{e^{-z^2}}{(-(z-x))^{n+1}} (-dz) \right)$$

(where we have made the change of variable $z = x - h$ and γ' is a simple closed curve about h)

$$= \sum_{n=0}^{\infty} \frac{h^n}{n!} (-1)^n e^{x^2} \left(\frac{n!}{2\pi i} \int_{\gamma'} \frac{e^{-h^2}}{(h-x)^{n+1}} dh \right)$$

(where we have renamed the dummy variable z by h)

$$= \sum_{n=0}^{\infty} \frac{h^n}{n!} (-1)^n e^{x^2} \left(\frac{d^n}{dh^n} e^{-h^2} \bigg|_{h=x} \right),$$

where we have used the Cauchy Integral Formula once again. So, we now have two Taylor series representations about 0 for $f(h, x)$; thus, by the uniqueness of Taylor series, we must have

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2},$$

as desired. $\square$