$\boxed{\text{Ex. 1}}$ Show that the delta function $\delta(x)$ can be expressed as

(a)

$$\delta(x) = \frac{1}{\pi} \lim_{N\to\infty} \frac{\sin Nx}{x};$$

(b)

$$\delta(x) = \frac{1}{2} \frac{d^2}{dx^2}|x|.$$

Proof. (a) There are many ways to prove this result- one can do it rather directly using the Riemann-Lebesgue lemma, or a little more indirectly in the context of distribution theory by showing the sequence of functions present in the limit are an approximation to the identity. We, however, will prove it using machinery more naturally occurring within quantum mechanics- the theory of the Fourier transform. To do this we will require the power of the Fourier Inversion Theorem, given below without proof. The interested reader can find a proof in any Fourier analysis text, or any modern treatment of real analysis (or, to put it within the correct context, a book on distribution theory to see the suitable generalization to tempered distributions, cf [1], page 95).

Theorem. *Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ (i.e. ϕ is a smooth function on $\mathbb{R}^n$ which is rapidly decreasing). If the Fourier transform of ϕ is defined as*

$$\widetilde{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x)e^{-i\xi\cdot x}dx,$$

then

$$\phi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widetilde{\phi}(\xi)e^{i\xi\cdot x}d\xi.$$

Before proving the claim, we will consider the Fourier transform of $\delta(x)$ and what the above theorem then implies. We have

$$\widetilde{\delta}(\xi) = \int_{\mathbb{R}} \delta(x)e^{-i\xi x}dx = e^{-i\xi(0)} = 1.$$

Thus, by the Fourier Inversion Theorem

$$\delta(x) = (2\pi)^{-1} \int_{\mathbb{R}} \widetilde{\delta}(\xi)e^{i\xi x}d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}d\xi.$$

But we can rewrite this improper integral as a limit:

$$\delta(x) = \frac{1}{2\pi} \lim_{N\to\infty} \int_{-N}^{N} e^{i\xi x}d\xi = \frac{1}{2\pi} \lim_{N\to\infty} \left.\frac{e^{i\xi x}}{ix}\right|_{-N}^{N}$$

$$= \frac{1}{2\pi} \lim_{N\to\infty} \frac{e^{iNx} - e^{-iNx}}{ix} = \frac{1}{\cancel{2}\pi} \lim_{N\to\infty} \frac{\cancel{2}\sin(Nx)}{x};$$

hence

$$\delta(x) = \frac{1}{\pi} \lim_{N\to\infty} \frac{\sin Nx}{x}.$$

(b) This follows from a couple of fairly simple distributional calculations. First, we define the *sign function*, $\operatorname{sgn}(x)$:

$$\operatorname{sgn}(x) = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

(Note: in a slight abuse of notation, we will also denote the distribution associated with $\operatorname{sgn}(x)$ by the same symbol when no confusion will arise.) The first step is to verify that, as distributions on $\mathbb{R}$ (although, in light of the probable unfamiliarity of the reader with distribution theory, we will not make use of the definition of the derivative of a distribution, and simply do the calculation directly), we have $\frac{d}{dx}|x| = \operatorname{sgn}(x)$. Indeed, given $\phi \in C_c^\infty(\mathbb{R})$

$$\left\langle \frac{d}{dx}|x|, \phi \right\rangle = \int_{-\infty}^{\infty} \frac{d}{dx}|x|\phi(x)dx = \cancel{|x|\phi(x)|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} |x|\phi'(x)dx$$

$$= -\left(-\int_{-\infty}^{0} x\phi'(x)dx + \int_{0}^{\infty} x\phi'(x)dx\right) = \int_{-\infty}^{0} x\phi'(x)dx - \int_{0}^{\infty} x\phi'(x)dx$$

$$= \left(\cancel{x\phi(x)|_{-\infty}^{0}} - \int_{-\infty}^{0} \phi(x)dx\right) - \left(\cancel{x\phi(x)|_{0}^{\infty}} - \int_{0}^{\infty} \phi(x)dx\right)$$

$$= \int_{-\infty}^{0} (-1)\phi(x)dx + \int_{0}^{\infty} (1)\phi(x)dx = \langle \operatorname{sgn}(x), \phi\rangle,$$

where the boundary terms all vanish since ϕ has compact support; hence, they are equal as distributions. Taking this computation one more step gives us what we want (where we will use the definition of distributional derivatives for the sake of brevity):

$$\left\langle \frac{d^2}{dx^2}|x|, \phi \right\rangle = -\left\langle \frac{d}{dx}|x|, \phi' \right\rangle = -\langle \operatorname{sgn}(x), \phi'\rangle$$

$$= -\left(\int_{-\infty}^{0} (-1)\phi'(x)dx + \int_{0}^{\infty} (1)\phi'(x)dx\right) = (\phi(x)|_{-\infty}^{0}) + (-\phi(x)|_{0}^{\infty})$$

$$= (\phi(0) - \cancel{\lim_{t\to-\infty} \phi(t)}) + (-\cancel{\lim_{t\to\infty} \phi(t)} - (-\phi(0))) = 2\phi(0) = 2\langle \delta, \phi\rangle,$$

which (after dividing by 2) establishes the claim (distributionally).

$\square$

$\boxed{\text{Ex. 2}}$ Show that

$$\langle x| \mathbf{P}^2 |x'\rangle = \left(\frac{\hbar}{i}\right)^2 \delta''(x - x').$$

Generalize this in the following way

$$\langle x| F(\mathbf{P}) |x'\rangle = F\left(\frac{\hbar}{i}\frac{d}{dx}\right) \delta(x - x').$$

Proof. For the first claim we have that

$$\langle x| \mathbf{P}^2 |x'\rangle = \int_{\mathbb{R}} dx'' \langle x| \mathbf{P} |x''\rangle \langle x''| \mathbf{P} |x'\rangle$$

$$= \int_{\mathbb{R}} dx'' \left(\left(\frac{\hbar}{i}\right) \delta'(x - x'')\right) \left(\left(\frac{\hbar}{i}\right) \delta'(x'' - x')\right)$$

$$= \left(\frac{\hbar}{i}\right)^2 \int_{\mathbb{R}} dx''\delta'(x - x'')\delta'(x'' - x').$$

Allowing ourselves to be a little careless, and treating δ' and δ' like functions we let

$$\begin{aligned} u &= \delta'(x'' - x) \qquad & dv &= \delta'(x - x'')dx'' \\ \Rightarrow \quad du &= \delta''(x'' - x)dx'' \qquad & v &= -\delta(x - x'') \end{aligned}$$

and perform integration by parts to obtain

$$\langle x| \mathbf{P}^2 |x'\rangle = \left(\frac{\hbar}{i}\right)^2 \left[\cancel{-\delta(x - x'')\delta'(x'' - x')|_{-\infty}^{\infty}} - \int_{\mathbb{R}} dx''(-\delta''(x'' - x'))\delta(x - x'')\right]$$

$$= \left(\frac{\hbar}{i}\right)^2 \delta''(x - x'),$$

where the boundary terms disappeared since both δ and δ' are 0 away from $x < \infty$ and $x' < \infty$, repsectively. We can now generalize this to any power of $\mathbf{P}$ by induction in the following way:

$$\langle x| \mathbf{P}^n |x'\rangle = \left(\frac{\hbar}{i}\right)^n \delta^{(n)}(x - x'),$$

where $\delta^{(n)}$ is the n^{th} distributional derivative of the δ function. The argument is essentially identical to that given above in the case $n = 2$. Suppose that the above relation holds for $n \leq k$ and consider when $n = k + 1$:

$$\langle x| \mathbf{P}^{k+1} |x'\rangle = \int_{\mathbb{R}} dx'' \langle x| \mathbf{P} |x''\rangle \langle x''| \mathbf{P}^k |x'\rangle$$

$$= \int_{\mathbb{R}} dx'' \left(\left(\frac{\hbar}{i} \right) \delta'(x - x'') \right) \left(\left(\frac{\hbar}{i} \right)^k \delta^{(k)}(x'' - x') \right)$$
$$= \left(\frac{\hbar}{i} \right)^{k+1} \int_{\mathbb{R}} dx'' \delta'(x - x'') \delta^{(k)}(x'' - x').$$

Again treating δ' and $\delta^{(k)}$ like functions we let

$$\begin{aligned} u &= \delta^{(k)}(x'' - x) & dv &= \delta'(x - x'')dx'' \\ \Rightarrow \quad du &= \delta^{(k+1)}(x'' - x)dx'' & v &= \delta(x - x'') \end{aligned}$$

and perform integration by parts to obtain

$$\langle x | \mathbf{P}^{k+1} | x' \rangle = \left(\frac{\hbar}{i} \right)^{k+1} \left[\cancel{-\delta(x - x'')\delta^{(k)}(x'' - x')|_{-\infty}^{\infty}} - \int_{\mathbb{R}} dx'' (-\delta^{(k+1)}(x'' - x'))\delta(x - x'') \right]$$
$$= \left(\frac{\hbar}{i} \right)^{k+1} \delta^{(k+1)}(x - x'),$$

which establishes the claim by induction on n. Now, given a (sufficiently nice) function F, let the Taylor series about 0 for F be given by

$$F(X) = a_0 + a_1 X + a_2 X^2 + \cdots = \sum_{n=0}^{\infty} a_n X^n.$$

Then we have

$$\langle x | F(\mathbf{P}) | x' \rangle = \sum_{n=0}^{\infty} a_n \langle x | \mathbf{P}^n | x' \rangle = \sum_{n=0}^{\infty} a_n \left(\frac{\hbar}{i} \right)^n \delta^{(n)}(x - x')$$
$$= \sum_{n=0}^{\infty} a_n \left(\frac{\hbar}{i} \right)^n \left(\frac{d}{dx} \right)^n \delta(x - x') = \left(\sum_{n=0}^{\infty} a_n \left(\frac{\hbar}{i} \frac{d}{dx} \right)^n \right) \delta(x - x') = F\left(\frac{\hbar}{i} \frac{d}{dx} \right) \delta(x - x'),$$

as claimed. □

$\boxed{\text{Ex. 3}}$ Consider a quantum system with state $|\psi\rangle$ that is translated by a distance ξ by the unitary operator $\mathbf{U}(\mathbf{P}, \xi) = \exp(-i\xi\mathbf{P}/\hbar)$, i.e.

$$|\psi'\rangle = \mathbf{U}(\mathbf{P}, \xi) = e^{-\frac{i}{\hbar}\xi\mathbf{P}} |\psi\rangle .$$

Find the q-representation of $|\psi'\rangle$, which is the wave function $\psi'(x) = \langle x | \psi' \rangle = \langle x | \mathbf{U}(\mathbf{P}, \xi) | \psi \rangle$, explicitly and interpret the result.

<u>**Solution.**</u>

We have that

$$\langle x|\,\mathbf{U}(\mathbf{P};\xi)\,|\psi\rangle = \int_{-\infty}^{\infty} dx'\,\langle x|\,\mathbf{U}(\mathbf{P};\xi)\,|x'\rangle\,\langle x'\,|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx'\,\langle x|\left(\mathbf{I} - \frac{i}{\hbar}\xi\mathbf{P}\right)|x'\rangle\,\langle x'\,|\psi\rangle = \int_{-\infty}^{\infty} dx'\,\langle x|\,\mathbf{I}\,|x'\rangle\,\psi(x') - \frac{i}{\hbar}\xi\int_{-\infty}^{\infty} dx'\,\langle x|\,\mathbf{P}\,|x'\rangle\,\psi(x')$$

$$= \int_{-\infty}^{\infty} dx'\delta(x-x')\psi(x') - \frac{\cancel{i}}{\cancel{\hbar}}\xi\int_{-\infty}^{\infty} dx'\frac{\cancel{\hbar}}{\cancel{i}}\delta'(x-x')\psi(x')$$

$$= \psi(x) + \xi\psi'(x).$$

References

[1] G. Friedlander and M. Joshi. *Introduction to the Theory of Distributions*. Cambridge University Press, 1998.