Ex. 1 Prove that if A and B that both commute with their commutator, then

$$
e^{\mathbf{A}} e^{\mathbf{B}}=\exp \left(\mathbf{A}+\mathbf{B}+\frac{[\mathbf{A}, \mathbf{B}]}{2}\right)
$$

To prove this relation we will first require the following properties of commutators and exponentials of operators.

Lemma 1. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are operators which both commute with their commutator, $n \geq 2$, and $\lambda \in \mathbb{C}$, then
(1)

$$
\left[\boldsymbol{A}, \boldsymbol{B}^{n}\right]=n \boldsymbol{B}^{n-1}[\boldsymbol{A}, \boldsymbol{B}] ;
$$

$$
\begin{equation*}
\left[\boldsymbol{A}, e^{\lambda \boldsymbol{B}}\right]=\lambda[\boldsymbol{A}, \boldsymbol{B}] e^{\lambda \boldsymbol{B}} \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
e^{\lambda \boldsymbol{A}}[\boldsymbol{A}, \boldsymbol{B}]=[\boldsymbol{A}, \boldsymbol{B}] e^{\lambda \boldsymbol{A}} \quad \text { and } \quad e^{\lambda \boldsymbol{B}}[\boldsymbol{A}, \boldsymbol{B}]=[\boldsymbol{A}, \boldsymbol{B}] e^{\lambda \boldsymbol{B}} ; \tag{3}
\end{equation*}
$$

$$
\begin{equation*}
\exp \left(\frac{1}{2}\left[\boldsymbol{A}+\boldsymbol{B}, \frac{[\boldsymbol{A}, \boldsymbol{B}]}{2}\right]\right)=\boldsymbol{I} \tag{6}
\end{equation*}
$$

$$
\begin{equation*}
e^{\boldsymbol{A}+\boldsymbol{B}}=e^{\boldsymbol{A}} e^{\boldsymbol{B}} \exp \left(-\frac{[\boldsymbol{A}, \boldsymbol{B}]}{2}\right) \tag{7}
\end{equation*}
$$

Proof. (of Lemma 1)
(1) We will proceed by induction. To obtain this relation in the case when $n=2$, consider the fact that $\mathbf{B}$ commutes with the commutator of $\mathbf{A}$ and $\mathbf{B}$ :

$$
\begin{gathered}
\mathbf{B}[\mathbf{A}, \mathbf{B}]=[\mathbf{A}, \mathbf{B}] \mathbf{B} \\
\Leftrightarrow \quad \mathbf{B}(\mathbf{A B}-\mathbf{B A})=(\mathbf{A B}-\mathbf{B A}) \mathbf{B} \\
\Leftrightarrow \quad \mathbf{B A B}-\mathbf{B}^{2} \mathbf{A}=\mathbf{A} \mathbf{B}^{2}-\mathbf{B A B} \\
\Leftrightarrow \quad 2 \mathbf{B A B}=\mathbf{A B}^{2}+\mathbf{B}^{2} \mathbf{A}=\{\mathbf{A}, \mathbf{B}\},
\end{gathered}
$$

where $\{\cdot, \cdot\}$ denotes the anti-commutator. But this gives us that

$$
\left[\mathbf{A}, \mathbf{B}^{2}\right]=\{\mathbf{A}, \mathbf{B}\}-2 \mathbf{B}^{2} \mathbf{A}=2 \mathbf{B A B}-2 \mathbf{B}^{2} \mathbf{A}=2 \mathbf{B}(\mathbf{A} \mathbf{B}-\mathbf{B} \mathbf{A})=2 \mathbf{B}[\mathbf{A}, \mathbf{B}]
$$

as desired. Now, suppose that this relation is true for $n \leq k$ and consider the case when $n=k+1$ - the argument is starkly similar to the $n=2$ case. First, we note that upon repeated application of the fact that $\mathbf{B}$ commutes with $[\mathbf{A}, \mathbf{B}], \mathbf{B}^{k}$ also commutes with $[\mathbf{A}, \mathbf{B}]$. Thus, we have that

$$
\begin{gathered}
\mathbf{B}^{k}[\mathbf{A}, \mathbf{B}]=[\mathbf{A}, \mathbf{B}] \mathbf{B}^{k} \\
\Leftrightarrow \quad \mathbf{B}^{k}(\mathbf{A B}-\mathbf{B A})=(\mathbf{A B}-\mathbf{B} \mathbf{A}) \mathbf{B}^{k} \\
\Leftrightarrow \quad \mathbf{B}^{k} \mathbf{A B}-\mathbf{B}^{k+1} \mathbf{A}=\mathbf{A} \mathbf{B}^{k+1}-\mathbf{B A B}^{k} \\
\Leftrightarrow \quad \mathbf{B}^{k} \mathbf{A} \mathbf{B}+\mathbf{B A B}^{k}=\mathbf{A} \mathbf{B}^{k+1}+\mathbf{B}^{k+1} \mathbf{A}=\left\{\mathbf{A}, \mathbf{B}^{k+1}\right\} .
\end{gathered}
$$

Just as was the case when $n=2$, from here we can do a little manipulation to get:

$$
\begin{gathered}
{\left[\mathbf{A}, \mathbf{B}^{k+1}\right]=\left\{\mathbf{A}, \mathbf{B}^{k+1}\right\}-2 \mathbf{B}^{k+1} \mathbf{A}=\mathbf{B}^{k} \mathbf{A B}+\mathbf{B} \mathbf{A} \mathbf{B}^{k}-2 \mathbf{B}^{k+1} \mathbf{A}} \\
=\mathbf{B}\left(\mathbf{B}^{k-1} \mathbf{A B}+\mathbf{A B}^{k}-2 \mathbf{B}^{k} \mathbf{A}\right)=\mathbf{B}\left(\mathbf{B}^{k-1} \mathbf{A B}+\mathbf{A} \mathbf{B}^{k}-\mathbf{B}^{k} \mathbf{A}-\mathbf{B}^{k} \mathbf{A}\right) \\
=\mathbf{B}\left(\mathbf{B}^{k-1} \mathbf{A B}-\mathbf{B}^{k} \mathbf{A}+\mathbf{A} \mathbf{B}^{k}-\mathbf{B}^{k} \mathbf{A}\right)=\mathbf{B}\left(\mathbf{B}^{k-1}(\mathbf{A B}-\mathbf{B} \mathbf{A})+\left[\mathbf{A}, \mathbf{B}^{k}\right]\right) \\
\mathbf{B}\left(\mathbf{B}^{k-1}[\mathbf{A}, \mathbf{B}]+k \mathbf{B}^{k-1}[\mathbf{A}, \mathbf{B}]\right)=(k+1) \mathbf{B}^{k}[\mathbf{A}, \mathbf{B}],
\end{gathered}
$$

as desired. Thus, by induction, we have established the result for $n \geq 2$. (It is also vacuously true when $n=1$; however that case is not very interesting.)
(2) This follows from the definition of the exponential of an operator, as well as an application of (1). Indeed

$$
\begin{gathered}
{\left[\mathbf{A}, e^{\lambda \mathbf{B}}\right]=\mathbf{A} e^{\lambda \mathbf{B}}-e^{\lambda \mathbf{B}} \mathbf{A}} \\
=\mathbf{A}\left(\mathbf{I}+\lambda \mathbf{B}+\frac{\lambda^{2}}{2!} \mathbf{B}^{2}+\frac{\lambda^{3}}{3!} \mathbf{B}^{3}+\cdots\right)-\left(\mathbf{I}+\lambda \mathbf{B}+\frac{\lambda^{2}}{2!} \mathbf{B}^{2}+\frac{\lambda^{3}}{3!} \mathbf{B}^{3}+\cdots\right) \mathbf{A} \\
\left(\mathbf{A}+\lambda \mathbf{A B}+\frac{\lambda^{2}}{2!} \mathbf{A} \mathbf{B}^{2}+\frac{\lambda^{3}}{3!} \mathbf{A} \mathbf{B}^{3}+\cdots\right)-\left(\mathbf{A}+\lambda \mathbf{B} \mathbf{A}+\frac{\lambda^{2}}{2!} \mathbf{B}^{2} \mathbf{A}+\frac{\lambda^{3}}{3!} \mathbf{B}^{3} \mathbf{A}+\cdots\right) .
\end{gathered}
$$

By collecting the terms in powers of $\lambda$ we have

$$
\begin{gathered}
{\left[\mathbf{A}, e^{\lambda \mathbf{B}}\right]=\lambda^{0}(\mathbf{A}-\mathbf{A})} \\
+\lambda(\mathbf{A B}-\mathbf{B} \mathbf{A})+\frac{\lambda^{2}}{2!}\left(\mathbf{A} \mathbf{B}^{2}-\mathbf{B}^{2} \mathbf{A}\right)+\frac{\lambda^{3}}{3!}\left(\mathbf{A} \mathbf{B}^{3}-\mathbf{B}^{3} \mathbf{A}\right)+\cdots \\
= \\
=\lambda[\mathbf{A}, \mathbf{B}]+\frac{\lambda^{2}}{2!}\left[\mathbf{A}, \mathbf{B}^{2}\right]+\frac{\lambda^{3}}{3!}\left[\mathbf{A}, \mathbf{B}^{3}\right]+\cdots
\end{gathered}
$$

Notice that the type of terms showing up are precisely what we considered in (1). Thus, we have

$$
\begin{gathered}
{\left[\mathbf{A}, e^{\lambda \mathbf{B}}\right]=\lambda[\mathbf{A}, \mathbf{B}]+\frac{\lambda^{2}}{2!}(2 \mathbf{B}[\mathbf{A}, \mathbf{B}])+\frac{\lambda^{3}}{3!}\left(3 \mathbf{B}^{2}[\mathbf{A}, \mathbf{B}]\right)+\cdots} \\
\lambda[\mathbf{A}, \mathbf{B}]\left(\mathbf{I}+\lambda \mathbf{B}+\frac{\lambda^{2}}{2!} \mathbf{B}^{2}+\cdots\right)=\lambda[\mathbf{A}, \mathbf{B}] e^{\lambda \mathbf{B}}
\end{gathered}
$$

(3) This follows from a simple calculation using the definition of the exponential of an operator.

$$
\begin{gathered}
e^{\lambda \mathbf{A}}[\mathbf{A}, \mathbf{B}]=\left(\mathbf{I}+\lambda \mathbf{A}+\frac{\lambda^{2}}{2!} \mathbf{A}^{2}+\cdots\right)[\mathbf{A}, \mathbf{B}] \\
=[\mathbf{A}, \mathbf{B}]+\lambda \mathbf{A}[\mathbf{A}, \mathbf{B}]+\frac{\lambda^{2}}{2!} \mathbf{A}^{2}[\mathbf{A}, \mathbf{B}]+\cdots=[\mathbf{A}, \mathbf{B}]+[\mathbf{A}, \mathbf{B}] \mathbf{A}+\frac{\lambda^{2}}{2!}[\mathbf{A}, \mathbf{B}] \mathbf{A}^{2}+\cdots \\
=[\mathbf{A}, \mathbf{B}]\left(\mathbf{I}+\lambda \mathbf{A}+\frac{\lambda^{2}}{2!} \mathbf{A}^{2}+\cdots\right)=[\mathbf{A}, \mathbf{B}] e^{\lambda \mathbf{A}} .
\end{gathered}
$$

The argument for the second portion of the claim is completely analogous.
(4) This follows from a simple application of (1.38):

$$
\begin{aligned}
e^{-\lambda \mathbf{B}} e^{\lambda \mathbf{B}} & =e^{-\lambda \mathbf{B}} \mathbf{I} e^{\lambda \mathbf{B}}=\mathbf{I}+\lambda[\mathbf{A}, \mathbf{I}]+\frac{\lambda^{2}}{2!}[\mathbf{A},[\mathbf{A}, \mathbf{I}]]+\cdots \\
& =\mathbf{I}+\lambda \mathbf{O}+\frac{\lambda^{2}}{2!}[\mathbf{A},[\mathbf{A}, \mathbf{O}]]+\cdots=\mathbf{I} .
\end{aligned}
$$

(5) Note that this is similar to (1.38), but easier to derive from the above 4 properties. By premultiplying (2) by $e^{-\lambda \mathbf{B}}$ we obtain

$$
\begin{gathered}
e^{-\lambda \mathbf{B}}\left[\mathbf{A}, e^{\lambda \mathbf{B}}\right]=e^{-\lambda \mathbf{B}} \lambda[\mathbf{A}, \mathbf{B}] e^{\lambda \mathbf{B}} \\
e^{-\lambda \mathbf{B}}\left(\mathbf{A} e^{\lambda \mathbf{B}}-e^{\lambda \mathbf{B}} \mathbf{A}\right)=\lambda e^{-\lambda \mathbf{B}} e^{\lambda \mathbf{B}}[\mathbf{A}, \mathbf{B}] \\
\left.e^{-\lambda \mathbf{B}} \mathbf{A} e^{\lambda \mathbf{B}}-e^{-\lambda \mathbf{B}} e^{\lambda \mathbf{B}} \mathbf{A}\right)=\lambda[\mathbf{A}, \mathbf{B}] \\
\Rightarrow \quad e^{-\lambda \mathbf{B}} \mathbf{A} e^{\lambda \mathbf{B}}=\mathbf{A}+\lambda[\mathbf{A}, \mathbf{B}],
\end{gathered}
$$

as desired.
(6) This follows from a (relatively) simple computation and a few of the properties of the commutator given in the lecture notes. From (1.36a) followed by (1.36c) we have that

$$
\begin{aligned}
\exp ( & \left.\frac{1}{2}\left[\mathbf{A}+\mathbf{B}, \frac{[\mathbf{A}, \mathbf{B}]}{2}\right]\right)=\exp \left(-\frac{1}{2}\left[\frac{[\mathbf{A}, \mathbf{B}]}{2}, \mathbf{A}+\mathbf{B}\right]\right) \\
& =\exp \left(-\frac{1}{2}\left[\frac{[\mathbf{A}, \mathbf{B}]}{2}, \mathbf{A}\right]-\frac{1}{2}\left[\frac{[\mathbf{A}, \mathbf{B}]}{2}, \mathbf{B}\right]\right)
\end{aligned}
$$

Since both A and B commute with their commutator, we also have that

$$
\left[\frac{[\mathbf{A}, \mathbf{B}]}{2}, \mathbf{A}\right]=\left[\frac{[\mathbf{A}, \mathbf{B}]}{2}, \mathbf{B}\right]=\mathbf{O}
$$

hence

$$
\exp \left(\frac{1}{2}\left[\mathbf{A}+\mathbf{B}, \frac{[\mathbf{A}, \mathbf{B}]}{2}\right]\right)=e^{\mathbf{O}}=\mathbf{I}
$$

(7) This is really the heart of the argument for this exercise, and the proof is similar to that of (1.38). We begin by defining $f(\lambda)=e^{\lambda \mathbf{A}} e^{\lambda \mathbf{B}}$. Then we have that

$$
\begin{aligned}
f^{\prime}(\lambda) & =e^{\lambda \mathbf{A}} \mathbf{A} e^{\lambda \mathbf{B}}+e^{\lambda \mathbf{A}} e^{\lambda \mathbf{B}} \mathbf{B}=e^{\lambda \mathbf{A}} e^{\lambda \mathbf{B}} e^{-\lambda \mathbf{B}} \mathbf{A} e^{\lambda \mathbf{B}}+e^{\lambda \mathbf{A}} e^{\lambda \mathbf{B}} \mathbf{B} \\
& =e^{\lambda \mathbf{A}} e^{\lambda \mathbf{B}}\left(e^{-\lambda \mathbf{B}} \mathbf{A} e^{\lambda \mathbf{B}}+\mathbf{B}\right)=f(\lambda)\left(e^{-\lambda \mathbf{B}} \mathbf{A} e^{\lambda \mathbf{B}}+\mathbf{B}\right) .
\end{aligned}
$$

We can then apply (5) to the inside of the parentheses to get

$$
f^{\prime}(\lambda)=f(\lambda)(\mathbf{A}+\lambda[\mathbf{A}, \mathbf{B}]+\mathbf{B})
$$

From here we simply need to note that the solution to the above differential equation with $f(0)=1$ is

$$
f(\lambda)=e^{\lambda(\mathbf{A}+\mathbf{B})} \exp \left(\frac{\lambda^{2}}{2}[\mathbf{A}, \mathbf{B}]\right)
$$

Evaluating at $\lambda=1$ and moving the latter term to the other side gives the desired result.

Proof. (of Ex. 1) This essentially follows from (7). Indeed, (7) implies that

$$
\exp \left((\mathbf{A}+\mathbf{B})+\frac{[\mathbf{A}, \mathbf{B}]}{2}\right)=e^{\mathbf{A}+\mathbf{B}} \exp \left(\frac{[\mathbf{A}, \mathbf{B}]}{2}\right) \exp \left(-\frac{1}{2}\left[\mathbf{A}+\mathbf{B}, \frac{[\mathbf{A}, \mathbf{B}]}{2}\right]\right)
$$

however it follows from (6) that the latter term is I. Hence, applying (7) to the first term yields

$$
\exp \left((\mathbf{A}+\mathbf{B})+\frac{[\mathbf{A}, \mathbf{B}]}{2}\right)=e^{\mathbf{A}} e^{\mathbf{B}} \exp \left(-\frac{[\mathbf{A}, \mathbf{B}]}{2}\right) \exp \left(\frac{[\mathbf{A}, \mathbf{B}]}{2}\right)
$$

Finally, (4) tells us that the last two terms cancel, and we are left with

$$
\exp \left((\mathbf{A}+\mathbf{B})+\frac{[\mathbf{A}, \mathbf{B}]}{2}\right)=e^{\mathbf{A}} e^{\mathbf{B}}
$$

as desired.
Ex. 2 If observables $\mathbf{A}_{1}$ and $\mathbf{A}_{2}$ are not compatible, but their corresponding operators commute with the Hamiltonian operator $\mathbf{H}$, i.e.

$$
\left[\mathbf{A}_{1}, \mathbf{H}\right]=\left[\mathbf{A}_{2}, \mathbf{H}\right]=0
$$

show that the energy eigenstates are, in general, degenerate.
Proof. Since $\left[\mathbf{A}_{1}, \mathbf{H}\right]=\left[\mathbf{A}_{2}, \mathbf{H}\right]=0$, Proposition 7 tells us that there exist complete sets $\left\{\phi_{r}\right\}$ and $\left\{\psi_{s}\right\}$ so that

$$
\begin{array}{lll}
\mathbf{A}_{1} \phi_{r}=a_{r} \phi_{r} \\
\mathbf{H} \phi_{r}=E_{r} \phi_{r} & , & \mathbf{A}_{2} \psi_{s}=a_{s}^{\prime} \psi_{s} \\
\mathbf{H} \psi_{s}=E_{s}^{\prime} \psi_{s}
\end{array}
$$

Moreover, the completeness of $\left\{\psi_{s}\right\}$ implies that, for each $j$, there exist constants $C_{j, s}$ so that

$$
\phi_{j}=\sum_{s} C_{j, s} \psi_{s}
$$

Next, we will denote by $\operatorname{Pr}_{k}(\cdot)$ the projection operator along $\psi_{k}$, so that, in particular,

$$
\operatorname{Pr}_{k}\left(\phi_{j}\right)=C_{j, k} \psi_{k}
$$

Note that the $\left\{\operatorname{Pr}_{k}\left(\phi_{j}\right)\right\}_{k}$ also constitutes a complete set of energy eigenstates:

$$
\mathbf{H P r}_{k}\left(\phi_{j}\right)=\mathbf{H}\left(C_{j, k} \psi_{k}\right)=E_{k}^{\prime}\left(C_{j, k} \psi_{k}\right)=E_{k}^{\prime} \operatorname{Pr}_{k}\left(\phi_{j}\right)
$$

Completeness follows immediately from the definition of $\operatorname{Pr}_{k}(\cdot)$ and the completeness of $\left\{\psi_{s}\right\}$. Finally, we proceed by contradiction. Suppose that all of the energy eigenstates of $\mathbf{H}$ are non-degenerate, then $\left\{\mathbf{P r}_{k}\left(\phi_{j}\right)\right\}$ are pairwise orthogonal (by Proposition 2), and hence linearly independent. But this means $\left\{\operatorname{Pr}_{k}\left(\phi_{j}\right)\right\}$ is a complete set of linearly independent vectors which are simultaneously eigenvectors for $\mathbf{A}_{1}$ and $\mathbf{A}_{2}$ ! Indeed,

$$
\mathbf{A}_{2}\left(\operatorname{Pr}_{k}\left(\phi_{j}\right)\right)=\mathbf{A}_{2}\left(C_{j, k} \psi_{k}\right)=C_{j, k} a_{k}^{\prime} \psi_{k}=a_{k}^{\prime}\left(\mathbf{P r}_{k}\left(\phi_{j}\right)\right)
$$

and, using Proposition 6,

$$
\mathbf{A}_{1}\left(\operatorname{Pr}_{k}\left(\phi_{j}\right)\right)=\operatorname{Pr}_{k}\left(\mathbf{A}_{1} \phi_{j}\right)=a_{j}\left(\operatorname{Pr}_{k}\left(\phi_{j}\right)\right) .
$$

But this is precisely the definition of compatibility, contradicting the assumption that $\mathbf{A}_{1}$ and $\mathbf{A}_{2}$ are not compatible; therefore degeneracies must exist in the energy eigenstates.

Ex. 3 Consider a one-dimensional Hamiltonian

$$
\mathbf{H}=\frac{1}{2 m} \mathbf{P}^{2}+V(\mathbf{Q})
$$

and use the fact that the commutator of $\mathbf{Q}$ and $[\mathbf{Q}, \mathbf{H}]$ is a constant operator, to show that

$$
\sum_{k}\left(E_{k}-E_{s}\right)\left|Q_{s k}\right|^{2}=\frac{\hbar}{2 m}
$$

which is referred to as the Thomas-Reiche-Kuhn sum rule, where $Q_{s k}=\left(\psi_{s}, \mathbf{Q} \psi_{k}\right)$ and $\psi_{s}$ is the eigenstate of $\mathbf{H}$ with eigenvalue $E_{s}$, i.e. $\mathbf{H} \psi_{s}=E_{s} \psi_{s}$.
First we will introduce some notation, and then prove three lemmas. Given a vector $\phi \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, we can define a linear functional on $\mathcal{H}$, $f_{\phi}: \mathcal{H} \rightarrow \mathbb{C}$ (i.e. a $f_{\phi} \in \mathcal{H}^{*}$ ), by

$$
f_{\phi}(\psi)=(\phi, \psi)
$$

(In standard physics notation, $f_{\phi}$ is denoted by $\langle\phi|$ ). Then, given $\phi, \xi \in \mathcal{H}$, one can define the outer product of $\phi$ and $\xi, \mathbf{O p}(\phi, \xi)$, as the operator

$$
(\mathbf{O p}(\phi, \xi))(\psi)=\xi f_{\phi}(\psi)=(\phi, \psi) \xi
$$

Lemma 2. (Completeness Relation) Let $\left\{\phi_{i}\right\}$ be a complete set of orthonormal vectors. Then

$$
\sum_{i} \boldsymbol{O} \boldsymbol{p}\left(\phi_{i}, \phi_{i}\right)=\boldsymbol{I}
$$

Proof. (of Lemma 2) We simply check what happens when you evaluate this operator at an arbitrary vector $\psi$ :

$$
\left(\sum_{i} \mathbf{O} \mathbf{p}\left(\phi_{i}, \phi_{i}\right)\right)(\psi)=\sum_{i}\left(\mathbf{O} \mathbf{p}\left(\phi_{i}, \phi_{i}\right)\right)(\psi)=\sum_{i}\left(\phi_{i}, \psi\right) \phi_{i}=\psi
$$

where the final equality follows from the fact that $\left\{\phi_{i}\right\}$ is a complete orthonormal set. Hence, $\sum_{i} \mathbf{O p}\left(\phi_{i}, \phi_{i}\right) \equiv \mathbf{I}$.

Lemma 3. The commutator of $\boldsymbol{H}$ and $\boldsymbol{Q},[\boldsymbol{H}, \boldsymbol{Q}]$, is anti-Hermitian.
Proof. (of Lemma 3) Note that both $\mathbf{Q}$ and $\mathbf{H}$ are Hermitian. Indeed, using the properties of the adjoint developed in the previous homework assignment,

$$
\begin{aligned}
{[\mathbf{H}, \mathbf{Q}]^{\dagger} } & =(\mathbf{H Q}-\mathbf{Q H})^{\dagger}=(\mathbf{H Q})^{\dagger}-(\mathbf{Q} \mathbf{H})^{\dagger}=\mathbf{Q}^{\dagger} \mathbf{H}^{\dagger}-\mathbf{H}^{\dagger} \mathbf{Q}^{\dagger} \\
& =\mathbf{Q H}-\mathbf{H Q}=-(-\mathbf{Q} \mathbf{H}+\mathbf{H Q})=-[\mathbf{H}, \mathbf{Q}]
\end{aligned}
$$

Lemma 4. In the context of this exercise, we have

$$
\left(\psi_{k},[\boldsymbol{H}, \boldsymbol{Q}] \psi_{s}\right)=\left(E_{k}-E_{s}\right)\left(\psi_{k}, \boldsymbol{Q} \psi_{s}\right) .
$$

Proof. (of Lemma 4) This follows from a simple calculation, with a little maneuvering using the adjoint and the linearity and anti-linearity of the inner product in the second and first coordinates, respectively:

$$
\begin{gathered}
\left(\psi_{k},[\mathbf{H}, \mathbf{Q}] \psi_{s}\right)=\left(\psi_{k}, \mathbf{H} \mathbf{Q} \psi_{s}\right)-\left(\psi_{k}, \mathbf{Q H} \psi_{s}\right)=\left((\mathbf{H} \mathbf{Q})^{\dagger} \psi_{k}, \psi_{s}\right)-\left(\psi_{k}, \mathbf{Q} E_{s} \psi_{s}\right) \\
=\left(\mathbf{Q}^{\dagger} \mathbf{H}^{\dagger} \psi_{k}, \psi_{s}\right)-E_{s}\left(\psi_{k}, \mathbf{Q} \psi_{s}\right)=\left(\mathbf{H} \psi_{k}, \mathbf{Q} \psi_{s}\right)-E_{s}\left(\psi_{k}, \mathbf{Q} \psi_{s}\right) \\
=\left(E_{k} \psi_{k}, \mathbf{Q} \psi_{s}\right)-E_{s}\left(\psi_{k}, \mathbf{Q} \psi_{s}\right)=E_{k}^{*}\left(\psi_{k}, \mathbf{Q} \psi_{s}\right)-E_{s}\left(\psi_{k}, \mathbf{Q} \psi_{s}\right)=\left(E_{k}-E_{s}\right)\left(\psi_{k}, \mathbf{Q} \psi_{s}\right),
\end{gathered}
$$

where the final equality is justified by the fact that $E_{k}$ is an eigenvector of a Hermitian operator, and is therefore real.

Proof. (of Ex. 4) Now that we have all of that taken care of, the proof is rather straight-forward. First, the problem points out that the commutator of $\mathbf{Q}$ and $[\mathbf{H}, \mathbf{Q}]$ is constant, but we must find which constant. To do this, first consider the commutator itself:

$$
\begin{aligned}
& {[\mathbf{H}, \mathbf{Q}]=\left[\frac{1}{2 m} \mathbf{P}^{2}+V(\mathbf{Q}), \mathbf{Q}\right]=-\left[\mathbf{Q}, \frac{1}{2 m} \mathbf{P}^{2}+V(\mathbf{Q})\right]} \\
& =-\left[\mathbf{Q}, \frac{1}{2 m} \mathbf{P}^{2}\right]-[\mathbf{Q}, V(\mathbf{Q})]=-\frac{1}{2 m}\left[\mathbf{Q}, \mathbf{P}^{2}\right]=-\frac{i \hbar}{m} \mathbf{P}
\end{aligned}
$$

where the final equality follows form the following simple consequence of the 3 rd Postulate of Ouantum Mechanics:

$$
[\mathbf{Q}, F(\mathbf{P})]=i \hbar \frac{d}{d \mathbf{P}} F(\mathbf{P})
$$

But this means that

$$
[\mathbf{Q},[\mathbf{H}, \mathbf{Q}]]=-\frac{i \hbar}{m}[\mathbf{Q}, \mathbf{P}]=-\frac{i \hbar}{m}(i \hbar)=\frac{\hbar^{2}}{m}
$$

where we have, once again, used the third postulate. Finally we can establish the claim- for simplicity of notation, we will begin using $\mathbf{C}=[\mathbf{H}, \mathbf{Q}]$.

$$
\frac{\hbar^{2}}{m}=\left(\psi_{k}, \frac{\hbar^{2}}{m} \psi_{k}\right)=\left(\psi_{k},[\mathbf{Q}, \mathbf{C}] \psi_{k}\right)=\left(\psi_{k}, \mathbf{Q} \mathbf{C} \psi_{k}\right)-\left(\psi_{k}, \mathbf{C} \mathbf{Q} \psi_{k}\right)
$$

Then, using Lemma 2, we have

$$
\begin{gathered}
\frac{\hbar^{2}}{m}=\left(\psi_{k}, \mathbf{Q} \sum_{s} \mathbf{O p}\left(\psi_{s}, \psi_{s}\right) \mathbf{C} \psi_{k}\right)-\left(\psi_{k}, \mathbf{C} \sum_{s} \mathbf{O} \mathbf{p}\left(\psi_{s}, \psi_{s}\right) \mathbf{Q} \psi_{k}\right) \\
=\sum_{s}\left(\left(\psi_{k}, \mathbf{Q}\left(\mathbf{O} \mathbf{p}\left(\psi_{s}, \psi_{s}\right)\right)\left(\mathbf{C} \psi_{k}\right)\right)-\left(\psi_{k}, \mathbf{C}\left(\mathbf{O p}\left(\psi_{s}, \psi_{s}\right)\right)\left(\mathbf{Q} \psi_{k}\right)\right)\right) \\
=\sum_{s}\left(\left(\psi_{k}, \mathbf{Q}\left(\psi_{s}, \mathbf{C} \psi_{k}\right) \psi_{s}\right)-\left(\psi_{k}, \mathbf{C}\left(\psi_{s}, \mathbf{Q} \psi_{k}\right) \psi_{s}\right)\right) \\
=\sum_{s}\left(\left(\psi_{k}, \mathbf{Q} \psi_{s}\right)\left(\psi_{s}, \mathbf{C}, \psi_{k}\right)-\left(\psi_{k}, \mathbf{C} \psi_{s}\right)\left(\psi_{s}, \mathbf{Q} \psi_{k}\right)\right) \\
=\sum_{s}\left(\left(\psi_{k}, \mathbf{Q} \psi_{s}\right)\left(E_{s}-E_{k}\right)\left(\psi_{s}, \mathbf{Q} \psi_{k}\right)-\left(E_{k}-E_{s}\right)\left(\psi_{k}, \mathbf{Q} \psi_{s}\right)\left(\psi_{s}, \mathbf{Q} \psi_{k}\right)\right) \\
=2 \sum_{s}\left(E_{s}-E_{k}\right)\left(\psi_{k}, \mathbf{Q} \psi_{s}\right)\left(\psi_{s}, \mathbf{Q} \psi_{k}\right)
\end{gathered}
$$

Notice, however, that

$$
\left(\psi_{k}, \mathbf{Q} \psi_{s}\right)\left(\psi_{s}, \mathbf{Q} \psi_{k}\right)=\left(\psi_{s}, \mathbf{Q}^{\dagger} \psi_{k}\right)^{*}\left(\psi_{s}, \mathbf{Q} \psi_{k}\right)=\left(\psi_{s}, \mathbf{Q} \psi_{k}\right)^{*}\left(\psi_{s}, \mathbf{Q} \psi_{k}\right)=\left|\left(\psi_{s}, \mathbf{Q} \psi_{k}\right)\right|^{2}
$$

Putting this into the above, and dividing by 2 , we obtain

$$
\frac{\hbar^{2}}{2 m}=\sum_{s}\left(E_{s}-E_{k}\right)\left|Q_{s k}\right|^{2}
$$

Ex. 4 Let

$$
\mathbf{U}\left(L_{3}, \theta\right)=\exp \left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)\right]
$$

Prove that

$$
\mathbf{U} X \mathbf{U}^{-1}=X \cos \theta-Y \sin \theta
$$

and

$$
\mathbf{U} Y \mathbf{U}^{-1}=X \sin \theta+Y \cos \theta
$$

Proof. First we must determine $\mathbf{U}^{-1}$; however, in light of (4), it is clear that it is

$$
\mathbf{U}^{-1}\left(L_{3}, \theta\right)=\exp \left[-\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)\right]
$$

Next we will introduce some notation to simplify the expressions that follow. Let

$$
C_{X} \equiv C_{X}^{1}=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right), X\right]
$$

and define inductively

$$
C_{X}^{k+1}=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right), C_{X}^{k}\right]
$$

Then an application of (1.38) yields

$$
\begin{aligned}
\mathbf{U}^{-1} X \mathbf{U} & =\exp \left[-\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)\right] X \exp \left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)\right] \\
& =X+\frac{1}{1!} C_{X}^{1}+\frac{1}{2!} C_{X}^{2}+\cdots=X+\sum_{n=1}^{\infty} \frac{1}{n!} C_{X}^{n}
\end{aligned}
$$

Clearly, in order to understand this expression we must first examine the terms $C_{X}^{k}$. We will begin by computing explicitly the first few terms, and then generalizing from there. To do this, first recall the following expression for the momentum operator, as well as some commutation relations involving the momentum and position operators:

$$
P_{x}=\frac{\hbar}{i} \frac{\partial}{\partial x}, \quad P_{y}=\frac{\hbar}{i} \frac{\partial}{\partial y}
$$

and

$$
\left[X, P_{y}\right]=\left[Y, P_{x}\right]=0
$$

From these we have

$$
\begin{gathered}
C_{X}^{1}=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right), X\right]=\left(\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right) X\right)-\left(X \frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)\right) \\
=\frac{\not \partial \theta}{\hbar}\left(X \frac{\hbar}{\nless} \frac{\partial}{\partial y}(X)-Y \frac{\hbar}{\bar{\not}} \frac{\partial}{\partial x}(X)-X X \frac{\hbar}{\hbar} \frac{\partial}{\partial y}+X Y \frac{\hbar}{\nexists} \frac{\partial}{\partial x}\right) .
\end{gathered}
$$

By computing the partial derivatives, and using the commutation relation listed above, this becomes:

$$
C_{X}^{1}=\theta\left(X(0)-Y(1)-X \frac{\partial}{\partial y}(X)+X \frac{\partial}{\partial x}(Y)\right)=-\theta Y
$$

Similarly, we have that

$$
\begin{gathered}
C_{X}^{2}=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right), C_{X}^{1}\right]=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right),-\theta Y\right] \\
=\left(\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)(-\theta Y)\right)-\left((-\theta Y) \frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)\right) \\
=\frac{\not \theta^{2}}{\hbar}\left(X \frac{\hbar}{\nexists} \frac{\partial}{\partial y}(-Y)-Y \frac{\hbar}{\not x} \frac{\partial}{\partial x}(-Y)+Y X \frac{\hbar}{\not x} \frac{\partial}{\partial y}-Y Y \frac{\hbar}{\not x} \frac{\partial}{\partial x}\right) . \\
=\theta^{2}\left(X(-1)-Y(0)+Y \frac{\partial}{\partial y}(X)-Y \frac{\partial}{\partial x}(Y)\right)=-\theta^{2} X .
\end{gathered}
$$

From here we can prove by induction what happens in the general case, which is

$$
\begin{gathered}
C_{X}^{2 k}=(-1)^{k} \theta^{2 k} X \\
C_{X}^{2 k+1}=(-1)^{k+1} \theta^{2 k+1} Y
\end{gathered}
$$

Indeed, if we assume that it is true for $k \leq 2 j$, and consider the case when $k=$ $2 j+1$, then we have

$$
\begin{aligned}
& C_{X}^{2 j+1}=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right), C_{X}^{2 j}\right]=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right),(-1)^{j} \theta^{2 j} X\right] \\
= & \left(\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)\left((-1)^{j} \theta^{2 j} X\right)\right)-\left(\left((-1)^{j} \theta^{2 j} X\right) \frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)\right) \\
= & (-1)^{j} \frac{\hbar \theta^{2 j+1}}{\hbar}\left(X \frac{\hbar}{\nexists} \frac{\partial}{\partial y}(X)-Y \frac{\hbar}{\bar{\not}} \frac{\partial}{\partial x}(X)-X X \frac{\hbar}{\bar{\not}} \frac{\partial}{\partial y}+X Y \frac{\hbar}{\overline{\not x}} \frac{\partial}{\partial x}\right) . \\
= & (-1)^{j} \theta^{2 j+1}\left(X(0)-Y(1)+X \frac{\partial}{\partial y}(X)-X \frac{\partial}{\partial x}(Y)\right)=(-1)^{j+1} \theta^{2 j+1} Y .
\end{aligned}
$$

Similarly, for $k=2 j+2=2(j+1)$,

$$
\begin{aligned}
& C_{X}^{2(j+1)}=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right), C_{X}^{2 j+1}\right]=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right),(-1)^{j+1} \theta^{2 j+1} Y\right] \\
= & \left(\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)\left((-1)^{j+1} \theta^{2 j+1} Y\right)\right)-\left(\left((-1)^{j+1} \theta^{2 j+1} Y\right) \frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right)\right) \\
& =(-1)^{j+1} \frac{\hbar \theta^{2(j+1)}}{\hbar}\left(X \frac{\hbar}{\not x} \frac{\partial}{\partial y}(Y)-Y \frac{\hbar}{\bar{d}} \frac{\partial}{\partial x}(Y)-Y X \frac{\hbar}{\bar{d}} \frac{\partial}{\partial y}+Y Y \frac{\hbar}{\bar{d}} \frac{\partial}{\partial x}\right) . \\
= & (-1)^{j+1} \theta^{2(j+1)}\left(X(1)-Y(0)+Y \frac{\partial}{\partial y}(X)-Y \frac{\partial}{\partial x}(Y)\right)=(-1)^{j+1} \theta^{2(j+1)} X .
\end{aligned}
$$

Combining this with the cases where $k=1,2$ proven earlier, by induction the result is established. Now we will insert this information into the expression we are interested in:

$$
\begin{gathered}
\mathbf{U}^{-1} X \mathbf{U}=X+\sum_{n=1}^{\infty} \frac{1}{n!} C_{X}^{n}=X+\sum_{n=1}^{\infty} \frac{1}{(2 n!)} C_{X}^{2 n}+\sum_{n=0}^{\infty} \frac{1}{(2 n+1)!} C_{X}^{2 n+1} \\
=X+X \sum_{n=1}^{\infty} \frac{(-1)^{n}}{(2 n!)} \theta^{2 n}+Y \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2 n+1)!} \theta^{2 n+1} \\
=X\left(1+\sum_{n=1}^{\infty} \frac{(-1)^{n}}{(2 n!)} \theta^{2 n}\right)-Y \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n+1)!} \theta^{2 n+1} \\
\\
=X \cos \theta-Y \sin \theta,
\end{gathered}
$$

as desired. In a completely analogous fashion we can define

$$
C_{Y} \equiv C_{Y}^{1}=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right), Y\right]
$$

and

$$
C_{Y}^{k+1}=\left[\frac{i \theta}{\hbar}\left(X P_{y}-Y P_{x}\right), C_{Y}^{k}\right],
$$

and arrive at the general expressions

$$
\begin{aligned}
C_{Y}^{2 k} & =(-1)^{k} \theta^{2 k} Y \\
C_{Y}^{2 k+1} & =(-1)^{k} \theta^{2 k+1} X
\end{aligned}
$$

Putting this into the analogous application of (1.38) for $\mathbf{U}^{-1} Y \mathbf{U}$ we have

$$
\begin{gathered}
\mathbf{U}^{-1} Y \mathbf{U}=Y+\sum_{n=1}^{\infty} \frac{1}{n!} C_{Y}^{n}=Y+\sum_{n=1}^{\infty} \frac{1}{(2 n!)} C_{Y}^{2 n}+\sum_{n=0}^{\infty} \frac{1}{(2 n+1)!} C_{Y}^{2 n+1} \\
=Y+Y \sum_{n=1}^{\infty} \frac{(-1)^{n}}{(2 n!)} \theta^{2 n}+X \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n+1)!} \theta^{2 n+1} \\
=Y\left(1+\sum_{n=1}^{\infty} \frac{(-1)^{n}}{(2 n!)} \theta^{2 n}\right)+X \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n+1)!} \theta^{2 n+1} \\
\quad=Y \cos \theta+X \sin \theta .
\end{gathered}
$$

Ex. 5 Show that if

$$
\mathbf{U}(\vec{L}, \vec{\theta})=\exp \left(\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}\right)
$$

then $\mathbf{U}$ commutes with $\vec{X} \cdot \vec{X}, \vec{Y} \cdot \vec{Y}$, and $\vec{Z} \cdot \vec{Z}$. [Hint: Show that $\mathbf{U}\left(X^{2}+Y^{2}+Z^{2}\right) \mathbf{U}^{-1}=$ $X^{2}+Y^{2}+Z^{2}$.]

Proof. This is really similar to the previous exercise. First, recall the following expression for the angular momentum (cf [1], Chapter 2 for derivation):

$$
\vec{L}=\frac{\hbar}{i}\left(Y \frac{\partial}{\partial z}-Z \frac{\partial}{\partial y}, Z \frac{\partial}{\partial x}-X \frac{\partial}{\partial z}, X \frac{\partial}{\partial y}-Y \frac{\partial}{\partial x}\right)
$$

thus

$$
\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}=\theta_{1}\left(Y \frac{\partial}{\partial z}-Z \frac{\partial}{\partial y}\right)+\theta_{2}\left(Z \frac{\partial}{\partial x}-X \frac{\partial}{\partial z}\right)+\theta_{3}\left(X \frac{\partial}{\partial y}-Y \frac{\partial}{\partial x}\right)
$$

Just like the previous problem we will define

$$
C \equiv C^{1}=\left[\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}, X^{2}+Y^{2}+Z^{2}\right]
$$

and

$$
C^{k+1}=\left[\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}, C^{k}\right]
$$

Then we have that

$$
\begin{aligned}
& \frac{i}{\hbar} \vec{\theta} \cdot \vec{L}\left(X^{2}+Y^{2}+Z^{2}\right)=\theta_{1}\left(Y \frac{\partial}{\partial z}-Z \frac{\partial}{\partial y}\right)+\theta_{2}\left(Z \frac{\partial}{\partial x}-X \frac{\partial}{\partial z}\right)+\theta_{3}\left(X \frac{\partial}{\partial y}-Y \frac{\partial}{\partial x}\right)\left(X^{2}+Y^{2}+Z^{2}\right) \\
& \quad=\theta_{1}(Y(2 Z)-Z(2 Y))+\theta_{2}(Z(2 X)-X(2 Z))+\theta_{3}(X(2 Y)-Y(2 X))=\vec{\theta} \cdot \overrightarrow{0}=0
\end{aligned}
$$

Just as was the case in the previous exercise, the commutation relations ensure us that the other terms which appear in the commutator will vanish, i.e.

$$
\begin{aligned}
\left(X^{2}+Y^{2}+Z^{2}\right)\left(\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}\right)=( & \left.X^{2}+Y^{2}+Z^{2}\right)\left(\theta_{1}\left(Y \frac{\partial}{\partial z}-Z \frac{\partial}{\partial y}\right)+\theta_{2}\left(Z \frac{\partial}{\partial x}-X \frac{\partial}{\partial z}\right)\right. \\
& \left.+\theta_{3}\left(X \frac{\partial}{\partial y}-Y \frac{\partial}{\partial x}\right)\right)=0
\end{aligned}
$$

Therefore,

$$
C \equiv C^{1}=\left[\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}, X^{2}+Y^{2}+Z^{2}\right]=0
$$

Consequently,

$$
C^{k+1}=\left[\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}, C^{k}\right]=\left[\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}, 0\right]=0
$$

Thus, by a quick application of (1.38), much in the spirit of the previous exercise, we have

$$
\begin{gathered}
\mathbf{U}\left(X^{2}+Y^{2}+Z^{2}\right) \mathbf{U}^{-1}=\exp \left[-\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}\right]\left(X^{2}+Y^{2}+Z^{2}\right) \exp \left[\frac{i}{\hbar} \vec{\theta} \cdot \vec{L}\right] \\
=\left(X^{2}+Y^{2}+Z^{2}\right)+\sum_{n=1}^{\infty} \frac{1}{n!} C_{X}^{n}=X^{2}+Y^{2}+Z^{2}
\end{gathered}
$$

Thus, $\mathbf{U}$ commutes with $X^{2}+Y^{2}+Z^{2}$, and so it also commutes with individual term simply by the distributivity of operators.

## References

[1] A.R. Edmonds. Angular Momentum in Quantum Mechanics. Princeton University Press, 1957.

