

Mathematical Structure  
of

Quantum Mechanics.

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3.  $F(h, x) = e^{2hx - h^2} = e^{x^2} \cdot e^{-x^2 + 2hx - h^2} = e^{x^2} \cdot e^{-(x-h)^2}$

$$\therefore F(h, x) = \sum_n \frac{1}{n!} \cdot h^n \cdot e^{x^2} \left[ \frac{\partial^n}{\partial h^n} (e^{-(x-h)^2}) \right]_{h=0}$$

So I want to prove that  $\left. \frac{\partial^n}{\partial h^n} (e^{-(x-h)^2}) \right|_{h=0} = (-1)^n \cdot \left( \frac{d}{dx} \right)^n \cdot e^{-x^2}$

By induction, for  $n=1$ :

$$\left. \frac{\partial}{\partial h} (e^{-(x-h)^2}) \right|_{h=0} = -2(x-h) \cdot (-1) \cdot e^{-(x-h)^2} \Big|_{h=0}$$
$$= +2x e^{-x^2}$$
$$(-1) \frac{d}{dx} e^{-x^2} = 2x e^{-x^2}$$

$\Rightarrow$  correct!

We assume that equality holds for  $n=k$ .

For  $n=k+1$ :

$$\frac{d}{dx} \left( \frac{d}{dx} \right)^n e^{-x^2}$$
$$= \frac{d}{dx} \left( \left. \frac{\partial^n}{\partial h^n} (e^{-(x-h)^2}) \right|_{h=0} \right) \cdot (-1)^n$$
$$= \frac{\partial^n}{\partial h^n} \left( \left. \frac{\partial}{\partial x} e^{-(x-h)^2} \right|_{h=0} \right) \cdot (-1)^n$$
$$= \frac{\partial^n}{\partial h^n} \left( \left. -2(x-h) e^{-(x-h)^2} \right|_{h=0} \right) \cdot (-1)^n$$
$$= (-1)^{n+1} \frac{\partial^n}{\partial h^n} \frac{\partial}{\partial h} e^{-(x-h)^2} \Big|_{h=0} = (-1)^{n+1} \frac{\partial^{n+1}}{\partial h^{n+1}} e^{-(x-h)^2} \Big|_{h=0}$$

$$\therefore \left. \frac{\partial^n}{\partial h^n} e^{-(x-h)^2} \right|_{h=0} = (-1)^n \left( \frac{d}{dx} \right)^n e^{-x^2}$$

That is to say  $F(h, x) = \sum_n \frac{1}{n!} h^n \cdot e^{+x^2} \left[ \frac{\partial^n}{\partial h^n} e^{-(x-h)^2} \right]_{h=0}$

$$= \sum_n \frac{1}{n!} h^n \cdot e^{x^2} \cdot (-1)^n \left( \frac{d}{dx} \right)^n e^{-x^2}$$

$$= \sum_n \frac{1}{n!} h^n \cdot H_n(x)$$

$$\Rightarrow H_n(x) = (-1)^n \exp(+x^2) \left( \frac{d}{dx} \right)^n \exp(-x^2) \quad *$$

2.  $\langle 0 | X^2 | 0 \rangle = \frac{\hbar}{m\omega} \langle 0 | \zeta^+ | 0 \rangle$

$$= \frac{\hbar}{2m\omega} \langle 0 | a^+ a^+ + a^+ a + a a^+ + a a | 0 \rangle$$

$$= \frac{\hbar}{2m\omega} \langle 0 | a a^+ | 0 \rangle = \frac{\hbar}{2m\omega} \langle 1 | 1 \rangle = \frac{\hbar}{2m\omega}$$

$$\therefore \exp\left[-\frac{k^2}{2} \langle 0 | X^2 | 0 \rangle\right] = \exp\left[-\frac{k^2}{4} \cdot \frac{\hbar}{m\omega}\right]$$

$$\langle 0 | e^{ikx} | 0 \rangle = \int_{-\infty}^{\infty} e^{ik'\zeta} \cdot \frac{1}{\sqrt{\pi}} \cdot \exp(-\zeta^2) d\zeta, \text{ where } k' = k \cdot \sqrt{\frac{\hbar}{m\omega}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-\left(\zeta - \frac{ik'}{2}\right)^2\right] d\zeta \cdot \exp\left[-\frac{k'^2}{4}\right]$$

$$= \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} \cdot \exp\left[-\frac{k^2}{4} \cdot \frac{\hbar}{m\omega}\right] = \exp\left[-\frac{k^2}{4} \cdot \frac{\hbar}{m\omega}\right]$$

$$\Rightarrow \exp\left[-\frac{k^2}{2} \langle 0 | X^2 | 0 \rangle\right] = \langle 0 | e^{ikx} | 0 \rangle \quad *$$

$$1. \langle n | \hat{z} | n \rangle = \frac{1}{\sqrt{2}} \langle n | a^\dagger + a | n \rangle = 0$$

$$\begin{aligned} \langle n | \hat{z}^2 | n \rangle &= \frac{1}{2} \langle n | a^\dagger a^\dagger + a^\dagger a + a a^\dagger + a a | n \rangle \\ &= \frac{1}{2} \langle n | a a^\dagger | n \rangle = \frac{1}{2} (n+1) \end{aligned}$$

$$\therefore \Delta \hat{z} \text{ in } n \text{ particle state} = \sqrt{\frac{n+1}{2}}$$

$$\langle n | \hat{p}_z | n \rangle = 0$$

$$\begin{aligned} \langle n | \hat{p}_z^2 | n \rangle &= -\frac{1}{2} \langle n | a^\dagger a^\dagger - a^\dagger a - a a^\dagger + a a | n \rangle \\ &= \frac{1}{2} \langle n | a a^\dagger | n \rangle = \frac{n+1}{2} \end{aligned}$$

$$\therefore \Delta \hat{p}_z = \sqrt{\frac{n+1}{2}}$$

$$\Rightarrow \text{uncertainty } \Delta \hat{z} \Delta \hat{p}_z = \frac{n+1}{2} *$$