

Mathematical Structure
of
Quantum Mechanics

B98202037

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1. P-representation:

$$\left(\frac{p^2}{2m} + V(x) \right) |\psi\rangle = E |\psi\rangle$$

$$\Rightarrow \left(\frac{p^2}{2m} - E \right) \psi(p) = - \int \langle p | x' \rangle \langle x' | V(x) | x'' \rangle \langle x'' | p' \rangle \langle p' | \psi \rangle dx'' dx' dp'$$

$$= + \frac{g}{2\pi\hbar} \int e^{-\frac{i}{\hbar}(p-p')x'} \delta(x'-a) \psi(p') dx' dp'$$

$$= \frac{g}{2\pi\hbar} \int e^{-\frac{i}{\hbar}(p-p') \cdot a} \psi(p') dp'$$

$$\Rightarrow e^{+\frac{i}{\hbar}pa} \psi(p) = \frac{\frac{g}{2\pi\hbar}}{\frac{p^2}{2m} - E} \int e^{+\frac{i}{\hbar}p'a} \psi(p') dp'$$

$$\Rightarrow (\text{左右同時對 } p \text{ 積分}): \int e^{+\frac{i}{\hbar}pa} \psi(p) dp = \int \frac{\frac{g}{2\pi\hbar}}{\frac{p^2}{2m} - E} dp \cdot \int \psi(p') e^{+\frac{i}{\hbar}p'a} dp'$$

$$\therefore \int \frac{\frac{g}{2\pi\hbar}}{\frac{p^2}{2m} - E} dp = 1, \text{ let } \varepsilon = -E, \because E < 0 \therefore \varepsilon > 0.$$

$$\Rightarrow \int \frac{dp}{p^2 + 2m\varepsilon} = \frac{\pi\hbar}{mg}$$

$$\Rightarrow \frac{1}{\sqrt{2m\varepsilon}} \tan^{-1} \left(\frac{p}{\sqrt{2m\varepsilon}} \right) \Big|_{p=-\infty}^{p=\infty} = \frac{1}{\sqrt{2m\varepsilon}} \cdot \pi = \frac{\pi\hbar}{mg} \Rightarrow \varepsilon = \frac{mg^2}{2\hbar^2} \Rightarrow E = -\frac{mg^2}{2\hbar^2}$$

$$\therefore \psi(p) = \frac{e^{ipq}}{\frac{p^2}{2m} + \frac{mg^2}{2\hbar^2}} \cdot N, \quad \Rightarrow \frac{1}{N^2} = \int |\psi(p)|^2 dp = \int \frac{dp}{\left(\frac{p^2}{2m} + \frac{mg^2}{2\hbar^2} \right)^2} \Rightarrow \psi(p) = \underbrace{\sqrt{\frac{mg^3}{2\pi\hbar^3}}}_{\text{normalization constant}} \cdot \underbrace{\frac{e^{ipq/\hbar}}{\frac{p^2}{2m} + \frac{mg^2}{2\hbar^2}}}_{\star}$$

ψ -representation:

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} (E + q\delta(x-a)) \psi = 0$$

積分 $\int_{a-\varepsilon}^{a+\varepsilon} dx, \varepsilon \rightarrow 0^+$,

$$\text{則得 } \left. \frac{d\psi}{dx} \right|_{a+\varepsilon} - \left. \frac{d\psi}{dx} \right|_{a-\varepsilon} = - \frac{2mg}{\hbar^2} \psi(a) \quad \text{--- ①}$$

$$\text{在 } x \neq a \text{ 之區域}, \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2} \psi = 0$$

$$\text{let } \beta = \sqrt{-\frac{2mE}{\hbar^2}}, (\because E < 0)$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + (-\beta^2) \psi = 0 \Rightarrow \psi = A e^{\beta(x-a)} + B e^{-\beta(x-a)}$$

$\because \psi$ 以 a 為中心左右對稱, 且 $\psi \rightarrow 0$ as $x \rightarrow \infty$ or $-\infty$.

$$\begin{cases} \psi = C e^{-\beta(x-a)}, \text{ for } x > a \\ \psi = C e^{+\beta(x-a)}, \text{ for } x < a \end{cases} \Rightarrow \begin{cases} \psi' = -\beta C e^{-\beta(x-a)}, x > a \\ \psi' = \beta C e^{+\beta(x-a)}, x < a \end{cases}$$

$$\text{代入 ① 式, } -2\beta C = -\frac{2mg}{\hbar^2} C$$

$$\Rightarrow -\frac{2mE}{\hbar^2} = \frac{m^2 g^2}{\hbar^4} \Rightarrow E = \frac{-mg^2}{2\hbar^2} \quad *$$

$$\Rightarrow \psi(x) = \frac{\sqrt{mg}}{\hbar} e^{-\frac{(x-a) \cdot mg}{\hbar^2}} \quad *$$

/ 0

2. quantum state $|\psi\rangle$ must satisfy

$$\textcircled{1} \quad (p - \langle p \rangle) |\psi\rangle = c(x - \langle x \rangle) |\psi\rangle$$

$$\textcircled{2} \quad \langle \psi | (p - \langle p \rangle)(x - \langle x \rangle) + (x - \langle x \rangle)(p - \langle p \rangle) |\psi\rangle = 0$$

\textcircled{1} in p-representation:

$$(p - \langle p \rangle) \psi(p) = c \left(i\hbar \frac{d}{dp} - \langle x \rangle \right) \psi(p)$$

$$\Rightarrow (p - \langle p \rangle + c \langle x \rangle) \psi(p) = i\hbar c \cdot \frac{d \psi(p)}{dp}$$

$$\Rightarrow \frac{d \psi}{\psi} = \frac{i}{\hbar c} \left(\frac{p - \langle p \rangle}{c} + \langle x \rangle \right) dp$$

we can shift our origin $\langle p \rangle = 0$, i.e., in a new frame of reference

$$\langle p \rangle = 0.$$

$$\therefore \frac{d \psi}{\psi} = \frac{i}{\hbar c} (p + c \langle x \rangle) dp$$

$$\Rightarrow \psi = \psi(0) \cdot e^{i \langle x \rangle p / \hbar} \cdot e^{ip^2 / 2\hbar c}$$

$\because (p - \langle p \rangle) |\psi\rangle = c(x - \langle x \rangle) |\psi\rangle$ and $\langle \psi | (p - \langle p \rangle) = \langle \psi | (x - \langle x \rangle) c^*$,

$$\textcircled{2} \text{ becomes: } \langle \psi | \frac{1}{c} p^2 + \left(\frac{1}{c}\right)^* p^2 |\psi\rangle = 0$$

$$\Rightarrow \langle \psi | p^2 |\psi\rangle \cdot \left(\frac{1}{c} + \left(\frac{1}{c}\right)^* \right) = 0$$

so that $\frac{1}{c}$ must be pure imaginary, let $\frac{1}{c} = i\alpha$, α is real. *

$$\therefore \psi = \psi(0) \cdot e^{i \langle x \rangle p / \hbar} \cdot e^{-p^2 \alpha / 2\hbar}$$

Shift the origin back. Then we have:

$$\psi(p) = \psi(\langle p \rangle) e^{i\langle x \rangle (p - \langle p \rangle)/\hbar} \cdot e^{-(p - \langle p \rangle)^2/\Delta}, \text{ where } \Delta = \frac{2t}{2\pi}$$