

Math. Struct. of QM HW. 2 頁番 897507034

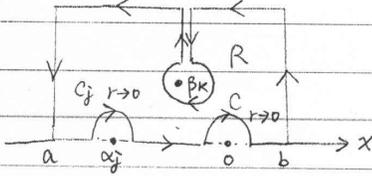
$$\text{Ex. 1 (a.) } \int_a^b dx \frac{1}{\pi} \lim_{N \rightarrow \infty} \frac{\sin NX}{x} f(x)$$

$$= \frac{1}{\pi} \text{Im} \lim_{N \rightarrow \infty} \int_a^b dx \frac{e^{iNX}}{x} f(x)$$

Suppose  $f(z)$  has singularities at  $\left\{ \begin{array}{l} z = \alpha_j, \text{Im}(\alpha_j) = 0, a < \alpha_j < b \\ z = \beta_k, \text{Im}(\beta_k) > 0 \end{array} \right.$

(I) For  $a < 0 < b$ ,

$$\lim_{N \rightarrow \infty} \int_a^b dx \frac{e^{iNX}}{x} f(x) = \lim_{N \rightarrow \infty} \left[ \oint - \int_R - \sum_j \int_{C_j} - \int_C \right] dz \frac{e^{iNz}}{z} f(z)$$



where  $\oint dz \frac{e^{iNz}}{z} f(z) = 0$

$$\lim_{N \rightarrow \infty} \int_R dz \frac{e^{iNz}}{z} f(z)$$

$$= \lim_{N \rightarrow \infty} \int_R dz \frac{1}{z} e^{iN \text{Re}(z)} e^{-N \text{Im}(z)} f(z), \text{Im}(z) > 0$$

$$= 0$$

$$\lim_{N \rightarrow \infty} \int_{C_j} dz \frac{e^{iNz}}{z} f(z)$$

$z = \alpha_j + re^{i\theta}, dz = ire^{i\theta} d\theta$

$$= \lim_{N \rightarrow \infty} \lim_{r \rightarrow 0} i \int_{\pi}^0 \frac{e^{iN(\alpha_j + re^{i\theta})}}{\alpha_j + re^{i\theta}} f(\alpha_j + re^{i\theta}) re^{i\theta} d\theta$$

$$= \lim_{N \rightarrow \infty} \lim_{r \rightarrow 0} i \int_{\pi}^0 \frac{re^{i\theta}}{\alpha_j + re^{i\theta}} e^{iN\alpha_j} f(\alpha_j + re^{i\theta}) d\theta$$

$$= 0$$

$$\lim_{N \rightarrow \infty} \int_C dz \frac{e^{iNz}}{z} f(z)$$

$z = re^{i\theta}, dz = ire^{i\theta} d\theta$

$$= \lim_{N \rightarrow \infty} \lim_{r \rightarrow 0} i \int_{\pi}^0 \frac{e^{iNre^{i\theta}}}{re^{i\theta}} f(re^{i\theta}) re^{i\theta} d\theta$$

$$= \lim_{N \rightarrow \infty} \lim_{r \rightarrow 0} i \int_{\pi}^0 e^{iNre^{i\theta}} f(re^{i\theta}) d\theta$$

$$= \lim_{N \rightarrow \infty} i \int_{\pi}^0 f(0) d\theta \quad \text{if } f(0) \text{ exists.}$$

$$= -i\pi f(0)$$

$$\Rightarrow \lim_{N \rightarrow \infty} \int_a^b dx \frac{e^{iNX}}{x} f(x) = i\pi f(0)$$

$$\Rightarrow \lim_{N \rightarrow \infty} \int_a^b dx \frac{\sin NX}{x} f(x) = \frac{1}{\pi} \text{Im}(i\pi f(0))$$

$$= f(0)$$

(II) For  $0 < a$  or  $b < 0$ ,

$$\lim_{N \rightarrow \infty} \int_a^b dx \frac{e^{iNX}}{x} f(x) = \lim_{N \rightarrow \infty} \left[ \oint - \int_R - \sum_j \int_{C_j} \right] dz \frac{e^{iNz}}{z} f(z) = 0$$

timeless

$$\Rightarrow \int_a^b dx \lim_{N \rightarrow \infty} \frac{1}{\pi} \frac{\sin Nx}{x} f(x)$$

$$= \begin{cases} f(0) & \text{if } a < 0 < b \text{ and if } f(0) \text{ exists.} \\ 0 & \text{if } 0 < a \text{ or } b < 0 \end{cases}$$

$$\therefore \lim_{N \rightarrow \infty} \frac{1}{\pi} \frac{\sin Nx}{x} = \delta(x)$$

(b) For  $0 < a$ ,

$$\int_a^b dx \frac{1}{2} \left( \frac{d}{dx} \right)^2 |x| f(x)$$

$$= \int_a^b dx \frac{1}{2} \left( \frac{d}{dx} \right)^2 x f(x)$$

$$= 0$$

For  $b < 0$ ,

$$\int_a^b dx \frac{1}{2} \left( \frac{d}{dx} \right)^2 |x| f(x)$$

$$= \int_a^b dx \frac{1}{2} \left( \frac{d}{dx} \right)^2 (-x) f(x)$$

$$= 0$$

For  $a < 0 < b$ ,

$$\int_a^b dx \frac{1}{2} \left( \frac{d}{dx} \right)^2 |x| f(x)$$

$$= \frac{1}{2} \frac{d}{dx} |x| f(x) \Big|_a^b - \int_a^b dx \frac{1}{2} \frac{d}{dx} |x| f'(x)$$

$$= \frac{1}{2} [f(b) + f(a)] - \int_a^0 dx \frac{1}{2} \frac{d}{dx} (-x) f'(x) - \int_0^b dx \frac{1}{2} \frac{d}{dx} (x) f'(x)$$

$$= \frac{1}{2} [f(b) + f(a)] + \int_a^0 dx \frac{1}{2} f'(x) - \int_0^b dx \frac{1}{2} f'(x)$$

$$= \frac{1}{2} [f(b) + f(a)] + \frac{1}{2} f(x) \Big|_a^0 - \frac{1}{2} f(x) \Big|_0^b$$

$$= \frac{1}{2} [f(b) + f(a) + f(0) - f(a) - f(b) + f(0)]$$

$$= f(0)$$

$$\therefore \frac{1}{2} \left( \frac{d}{dx} \right)^2 |x| = \delta(x)$$

$$\text{Ex. 2} \quad \langle x|P|x'\rangle = \frac{\hbar}{i} \frac{d}{dx} \delta(x-x') = \frac{\hbar}{i} \delta(x-x') \frac{d}{dx'}$$

$$\langle x|P^2|x'\rangle = \langle x|P \mathbb{I} P|x'\rangle$$

$$= \int dx'' \langle x|P|x''\rangle \langle x''|P|x'\rangle$$

$$= \int dx'' \frac{\hbar}{i} \delta(x-x'') \frac{d}{dx''} \frac{\hbar}{i} \frac{d}{dx''} \delta(x''-x')$$

$$= \left(\frac{\hbar}{i}\right)^2 \int dx'' \delta(x-x'') \left(\frac{d}{dx''}\right)^2 \delta(x''-x')$$

$$= \left(\frac{\hbar}{i}\right)^2 \left(\frac{d}{dx}\right)^2 \delta(x-x')$$

$$= \left(\frac{\hbar}{i}\right)^2 \delta''(x-x')$$

$$\langle x|P^n|x'\rangle = \langle x|P \mathbb{I} P^{n-1}|x'\rangle$$

$$= \int dx'' \langle x|P|x''\rangle \langle x''|P^{n-1}|x'\rangle$$

$$= \int dx'' \frac{\hbar}{i} \delta(x-x'') \frac{d}{dx''} \langle x''|P^{n-1}|x'\rangle$$

$$= \frac{\hbar}{i} \frac{d}{dx} \langle x|P^{n-1}|x'\rangle$$

$$= \left(\frac{\hbar}{i} \frac{d}{dx}\right)^n \delta(x-x')$$

$$\Rightarrow \langle x|F(P)|x'\rangle = F\left(\frac{\hbar}{i} \frac{d}{dx}\right) \delta(x-x')$$

for any function  $F(P)$  that can be expanded into a series of  $P^n$

$$\text{Ex. 3} \quad |\psi'\rangle = U(P, \xi) |\psi\rangle = e^{-\frac{i}{\hbar} \xi P} |\psi\rangle$$

$$\langle x|\psi'\rangle = \langle x|U|\psi\rangle$$

$$U X U^{-1} = e^{-\frac{i}{\hbar} \xi P} X e^{\frac{i}{\hbar} \xi P} = X + \left(-\frac{i}{\hbar} \xi\right) [P, X]$$

$$= X - \frac{i}{\hbar} \xi \left(\frac{\hbar}{i}\right) \mathbb{I} = X - \xi \mathbb{I}$$

$$\Rightarrow U X = X U - \xi U$$

$$\Rightarrow \langle x|U X = \langle x|(X - \xi)U = \langle x|(x - \xi)U = \langle x|U(x - \xi)$$

$$\therefore \langle x|U = \langle x - \xi|$$

$$\Rightarrow \langle x|\psi'\rangle = \langle x - \xi|\psi\rangle$$

$$= \psi(x - \xi)$$