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10 $\boxed{\text{Ex. 1}}$ Show that, for bounded operators $\mathbf{A}$ and $\mathbf{B}$, we have $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$.

Proof. This follows almost immediately from the definition of $\|\cdot\|$. For ψ in our vector space, since $\mathbf{A}$ and $\mathbf{B}$ are bounded:

$$\|\mathbf{AB}\psi\| \leq \|\mathbf{A}\|\|\mathbf{B}\psi\| \leq \|\mathbf{A}\|\|\mathbf{B}\|\|\psi\|.$$

Thus, given $x \neq 0$ we have

$$\frac{\|\mathbf{AB}\psi\|}{\|\psi\|} \leq \frac{\|\mathbf{A}\|\mathbf{B}\|\cancel{\|\psi\|}}{\cancel{\|\psi\|}} = \|\mathbf{A}\|\|\mathbf{B}\|.$$

Taking the supremum over all such vectors ψ on both sides we have

$$\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|,$$

as desired. □

10 $\boxed{\text{Ex. 2}}$ Let $\{\phi_1, \phi_2, \cdots, \phi_n\}$ be an orthonormal basis. Prove that the operator $\mathbf{U}$ is unitary if $\{\mathbf{U}\phi_1, \mathbf{U}\phi_2, \cdots, \mathbf{U}\phi_n\}$ is also an orthonormal basis.

Proof. Assume that $\{\mathbf{U}\phi_1, \cdots, \mathbf{U}\phi_n\}$ is orthonormal. Then, by definition, we have

$$(\mathbf{U}\phi_i, \mathbf{U}\phi_j) = \delta_{ij}.$$

But, by the definition of the adjoint, this is equivalent to

$$(\mathbf{U}^\dagger(\mathbf{U}\phi_i), \phi_j) = \delta_{ij} \quad \Leftrightarrow \quad (\mathbf{U}^\dagger\mathbf{U}\phi_i, \phi_j) = \delta_{ij}$$

Note that this leaves us with

$$(\phi_i, \phi_j) = \delta_{ij} = (\mathbf{U}^\dagger\mathbf{U}\phi_i, \phi_j),$$

which actually establishes the claim since the action of $\mathbf{U}^\dagger\mathbf{U}$ is identical to that of $\mathbf{I}$ on the basis; however, in the interest of completeness, we will explicitly demonstrate that this implies their action is the same on a pair of arbitrary vectors. To this end, we let $\psi = \sum_{i=1}^{n} a_i\phi_i$ and $\xi = \sum_{j=1}^{n} b_j\phi_j$ be arbitrary vectors in our vector space and consider the following inner product:

$$(\mathbf{U}^\dagger\mathbf{U}\psi, \xi) = \left(\mathbf{U}^\dagger\mathbf{U}\sum_{i=1}^{n} a_i\phi_i, \xi\right) = \left(\sum_{i=1}^{n} a_i\mathbf{U}^\dagger\mathbf{U}\phi_i, \xi\right) = \sum_{i=1}^{n} a_i^*\left(\mathbf{U}^\dagger\mathbf{U}\phi_i, \sum_{j=1}^{n} b_j\phi_j\right)$$

$$= \sum_{i=1}^{n} a_i^* \sum_{j=1}^{n} b_j (\mathbf{U}^\dagger \mathbf{U}\phi_i, \phi_j) = \sum_{i=1}^{n} a_i^* \sum_{j=1}^{n} b_j \delta_{ij} = \sum_{i=1}^{n} a_i^* \sum_{j=1}^{n} b_j (\phi_i, \phi_j)$$

$$= \sum_{i=1}^{n} a_i^* \left(\phi_i, \sum_{j=1}^{n} b_j \phi_j \right) = \left(\sum_{i=1}^{n} a_i \phi_i, \xi \right) = (\psi, \xi).$$

Since ψ and ξ are arbitrary, we have that

$$\mathbf{U}^\dagger \mathbf{U} = \mathbf{I},$$

so that $\mathbf{U}$ is unitary. □

10

$\boxed{\text{Ex. 3}}$ Show that

(a) $(\mathbf{A}+\mathbf{B})^\dagger = \mathbf{A}^\dagger + \mathbf{B}^\dagger$

(b) $(\alpha \mathbf{A})^\dagger = \alpha^* \mathbf{A}^\dagger$

(c) $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger$

(d) $(\mathbf{A}^\dagger)^\dagger = \mathbf{A}$

(e) $(\mathbf{A}^\dagger)^{-1} = (\mathbf{A}^{-1})^\dagger$

Proof. (a) Consider the following inner product:

$$((\mathbf{A}+\mathbf{B})^\dagger \phi, \psi) = (\phi, (\mathbf{A}+\mathbf{B})\psi) = (\phi, \mathbf{A}\psi + \mathbf{B}\psi) = (\phi, \mathbf{A}\psi) + (\phi, \mathbf{B}\psi)$$

$$= (\mathbf{A}^\dagger \phi, \psi) + (\mathbf{B}^\dagger \phi, \psi) = ((\mathbf{A}^\dagger + \mathbf{B}^\dagger)\phi, \psi).$$

Since ϕ and ψ are arbitrary, this gives us that

$$(\mathbf{A}+\mathbf{B})^\dagger = \mathbf{A}^\dagger + \mathbf{B}^\dagger.$$

(b) Again consider the following inner product with ϕ and ψ arbitrary:

$$((\alpha \mathbf{A})^\dagger \phi, \psi) = (\phi, (\alpha \mathbf{A})\psi) = (\phi, \alpha(\mathbf{A}\psi)) = \alpha(\phi, \mathbf{A}\psi) = \alpha(\mathbf{A}^\dagger \phi, \psi) = (\alpha^* \mathbf{A}^\dagger \phi, \psi)$$

$$\Rightarrow \quad (\alpha \mathbf{A}^\dagger = \alpha^* \mathbf{A}^\dagger.$$

(c) We will follow the same procedure as in the previous two steps:

$$((\mathbf{AB})^\dagger \phi, \psi) = (\phi, (\mathbf{AB})\psi) = (\phi, \mathbf{A}(\mathbf{B}\psi)) = (\mathbf{A}^\dagger \phi, \mathbf{B}\psi) = (\mathbf{B}^\dagger(\mathbf{A}^\dagger \phi), \psi) = ((\mathbf{B}^\dagger \mathbf{A}^\dagger)\phi, \psi)$$

$$\Rightarrow \quad (\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger.$$

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(d) Once again we can perform the same trick by using (1.3c), as well as the fact that for any $z \in \mathbb{C}$ we have $(z^*)^* = z$:

$$((\mathbf{A}^\dagger)^\dagger\phi, \psi) = (\phi, \mathbf{A}^\dagger\psi) = (\mathbf{A}^\dagger\psi, \phi)^* = (\psi, \mathbf{A}\phi)^* = [(\mathbf{A}\phi, \psi)^*]^* = (\mathbf{A}\phi, \psi)$$
$$\Rightarrow \quad (\mathbf{A}^\dagger)^\dagger = \mathbf{A}.$$

(e) This one we can do explicitly, without worrying about inner products, by utilizing part (c):

$$\mathbf{A}^\dagger(\mathbf{A}^{-1})^\dagger = (\mathbf{A}^{-1}\mathbf{A})^\dagger = \mathbf{I}^\dagger = \mathbf{I}.$$

Thus, by definition of the inverse operator,

$$(\mathbf{A}^\dagger)^{-1} = (\mathbf{A}^{-1})^\dagger.$$

□

/10 $\boxed{\text{Ex. 4}}$ Show that $\mathbf{P}_\mathcal{M}\mathbf{P}_{\mathcal{M}_\perp} = \mathbf{P}_{\mathcal{M}_\perp}\mathbf{P}_\mathcal{M} = \mathbf{O}$.

Proof. This follows from a straightforward calculation. Let $\psi = \psi_\mathcal{M} + \psi_{\mathcal{M}_\perp}$, then by definition

$$(\mathbf{P}_\mathcal{M}\mathbf{P}_{\mathcal{M}_\perp})(\psi) = \mathbf{P}_\mathcal{M}(\mathbf{P}_{\mathcal{M}_\perp}(\psi_\mathcal{M} + \psi_{\mathcal{M}_\perp})) = \mathbf{P}_\mathcal{M}(\psi_{\mathcal{M}_\perp}) = \mathbf{O},$$

and

$$(\mathbf{P}_{\mathcal{M}_\perp}\mathbf{P}_\mathcal{M})(\psi) = \mathbf{P}_{\mathcal{M}_\perp}(\mathbf{P}_\mathcal{M}(\psi_\mathcal{M} + \psi_{\mathcal{M}_\perp})) = \mathbf{P}_{\mathcal{M}_\perp}(\psi_\mathcal{M}) = \mathbf{O}.$$

□

/10 $\boxed{\text{Ex. 5}}$ Consider the Hilbert space spanned by a Hermitian operator $\mathbf{A}$.

(a) Prove that $\prod_a(\mathbf{A} - a\mathbf{I})$ is the null operator if $\mathbf{A}\psi_a = a\psi_a$.

(b) What is the significance of the operator $\prod_{a \neq a'} \dfrac{\mathbf{A} - a\mathbf{I}}{a' - a}$?

Proof. (a) First, we will show that $\prod_a(\mathbf{A} - a\mathbf{I})$ applied to any of the ψ_a's is 0:

$$\left[\prod_a(\mathbf{A} - a\mathbf{I})\right]\psi_a = \prod_a[(\mathbf{A} - a\mathbf{I})\psi_a] = \prod_a(\mathbf{A}\psi_a - a\psi_a) = \prod_a(a\psi_a - a\psi_a) = 0.$$

From here we simply need to note that, by the 2nd postulate of quantum mechanics, the set $\{\psi_a\}$ form a basis for the Hilbert space, and since $\prod_a(\mathbf{A} - a\mathbf{I}) \equiv \mathbf{O}$ on the basis and is linear, we must have

$$\prod_a(\mathbf{A} - a\mathbf{I}) = \mathbf{O}.$$

(b) The significance is that

$$\sum_{a'}\left(\prod_{a\neq a'}\frac{\mathbf{A}-a\mathbf{I}}{a'-a}\right)=\mathbf{I}.$$

To see this we will first fix an a' and consider the action of the operator associated with a' on ψ_b for some eigenvalue b such that $b \neq a'$:

$$\left(\prod_{a\neq a'}\frac{\mathbf{A}-a\mathbf{I}}{a'-a}\right)\psi_b=\prod_{a\neq a'}\left(\frac{\mathbf{A}-a\mathbf{I}}{a'-a}\psi_b\right)=\prod_{a\neq a'}\frac{\mathbf{A}\psi_b-a\psi_b}{a'-a}=\prod_{a\neq a'}\frac{b\psi_b-a\psi_b}{a'-a}=0,$$

where the final equality follows from the assumption that b belongs to the collection of all a with $a \neq a'$; hence on of the terms in the product will vanish. Now consider what happen to $\psi_{a'}$ for this fixed a':

$$\left(\prod_{a\neq a'}\frac{\mathbf{A}-a\mathbf{I}}{a'-a}\right)\psi_{a'}=\prod_{a\neq a'}\left(\frac{\mathbf{A}-a\mathbf{I}}{a'-a}\psi_{a'}\right)=\prod_{a\neq a'}\frac{\mathbf{A}\psi_{a'}-a\psi_{a'}}{a'-a}$$

$$=\prod_{a\neq a'}\frac{a'\psi_{a'}-a\psi_{a'}}{a'-a}=\psi_{a'}\prod_{a\neq a'}\frac{a'-a}{a'-a}=\psi_{a'}\prod_{a\neq a'}1=\psi_{a'}.$$

Thus, looking at this operator in terms of its matrix elements, it will have a 1 in the a', a'-th spot, and 0's everywhere else. Hence, upon adding it up over all eigenvalues a', the resulting sum of these operators will have 1's along the diagonal and 0's everywhere else.

□