

(b) This formula is due to Sylvester, which enable one to compute the projections directly.

\* The advantage of this formula is that one doesn't need to know the eigenvectors in order to find the projections.

\* Since  $A$  is a Hermitian operator, suppose it's represented by a  $n \times n$  matrix, and change eigenvalue.

$$A = \sum_j \lambda_j |\lambda_j\rangle \langle \lambda_j| \quad \begin{array}{l} a_i \text{ to } \lambda_i \text{ or } \lambda_j \\ i, j \in \{1, 2, \dots, n\} \end{array}$$

$$= \lambda_i |\lambda_i\rangle \langle \lambda_i| + \sum_{\substack{j \\ (i \neq j)}} \lambda_j |\lambda_j\rangle \langle \lambda_j|$$

$$= \lambda_i (I_{n \times n} - \sum_{\substack{j \\ (i \neq j)}} |\lambda_j\rangle \langle \lambda_j|)$$

$$+ \sum_{\substack{j \\ (i \neq j)}} \lambda_j |\lambda_j\rangle \langle \lambda_j|$$

$$= \lambda_i I + \sum_{\substack{j \\ (i \neq j)}} (\lambda_j - \lambda_i) |\lambda_j\rangle \langle \lambda_j|$$

$$(A - \lambda_i I) = \sum_{\substack{j \\ (i \neq j)}} (\lambda_j - \lambda_i) |\lambda_j\rangle \langle \lambda_j|$$

By the Mathematical induction we can show that

$$\prod_{i=1}^n (A - \lambda_i I) = (\lambda_n - \lambda_1)(\lambda_n - \lambda_2) \dots (\lambda_n - \lambda_{n-1}) |\lambda_n\rangle \langle \lambda_n|$$

$\because j=n, j \neq 1, 2, \dots, n-1$

$$\Rightarrow (A - \lambda_1 I)(A - \lambda_2 I) \dots (A - \lambda_{n-1} I)(A - \lambda_{n+1} I) \dots (A - \lambda_n I) \\ = (\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3) \dots (\lambda_2 - \lambda_{n-1}) (\lambda_2 - \lambda_n) \dots (\lambda_{n-1} - \lambda_n) \dots$$

$$\Rightarrow \prod_{\substack{i=1 \\ i \neq l}}^n (A - \lambda_i I) = \left[ \prod_{\substack{i=1 \\ i \neq l}}^n (\lambda_l - \lambda_i) \right] |\lambda_l\rangle \langle \lambda_l|$$

$$\Rightarrow |\lambda_l\rangle \langle \lambda_l| = \frac{\prod_{\substack{i=1 \\ i \neq l}}^n (A - \lambda_i I)}{\prod_{\substack{i=1 \\ i \neq l}}^n (\lambda_l - \lambda_i)} = \frac{\prod_{\substack{i=1 \\ i \neq l}}^n (A - \alpha_i I)}{\prod_{\substack{i=1 \\ i \neq l}}^n (\alpha_l - \alpha_i)}$$

Now, for a  $n \times n$  matrix  $A$  with distinct eigenvalues  $\lambda_j$  and a function  $f$ , so that the spectral decomposition law may be expressed as

$$P(A) = \sum_{j=1}^n f(\alpha_j) \frac{\prod_{i \neq j} (A - \alpha_i I)}{\prod_{i \neq j} (\alpha_j - \alpha_i)}$$

if we let

$$P(A) = \sum_i P(\alpha_i) |\lambda_i\rangle \langle \lambda_i|$$

where  $A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$ ,  $i=1, 2, \dots, n$

then

$$\begin{aligned} P(A) &= \sum_l P(\alpha_l) |\lambda_l\rangle \langle \lambda_l| \quad \text{--- ①} \\ &= \sum_l P(\alpha_l) \frac{\prod_{\substack{i=1 \\ i \neq l}}^n (A - \alpha_i I)}{\prod_{\substack{i=1 \\ i \neq l}}^n (\alpha_l - \alpha_i)} \quad \text{--- ②} \end{aligned}$$

and compare with ① and ② we conclude that we don't need to know eigenvectors when eigenvalues are distinct, but still can find the projections!